MATH 162 SOLUTIONS: TEST I 16 FEBRUARY 2018

Instructions: Answer any 16 of the following 18 questions. You may answer more than 16 to obtain extra credit. *You must show your reasoning*; calculator answers are not acceptable.
Each problem is worth 6 points, unless otherwise stated.

1. Explain why the following improper integral converges.

$$\int_1^{\infty} \frac{x^2-1}{2x^5+3x+17}\,dx$$

Solution: Since the integrand is roughly, $\frac{x^2}{2x^5} = \frac{1}{2}\ \frac{1}{x^3}$ *, we conjecture that our given integral converges (by virtue of the p-test for Type 1 improper integrals).*

Using the Comparison Test, for x ≥ 1, $0 \leq \frac{x^2-1}{2x^5+3x+17} < \frac{x^2}{x^5} = \frac{1}{x^3}$ *we see that our original integral converges.*

2. Determine if the following improper integral converges or diverges. *Justify your answer!*

$$\int_{5+}^{7} \frac{1}{\sqrt{x-5}}\,dx$$

Solution:

Since $\int_{5+}^{7} \frac{1}{\sqrt{x-5}}dx$ *appears to behave as* $\int_{0+}^{5} \frac{1}{\sqrt{x}}dx$*, we conjecture via the p-test for type 2 improper integrals that our original integral converges.*

Using the definition of the improper integral:

$$\int_{5+}^{7} \frac{1}{\sqrt{x-5}}dx = \lim_{a\to 5+} \int_{a}^{7} \frac{1}{\sqrt{x-5}}dx = \lim_{a\to 5+} 2(x-5)^{\frac{1}{2}}\Big|_a^7 =$$

$$\lim_{a\to 5+} 2(x-5)^{\frac{1}{2}}\Big|_a^7 = 2\lim_{a\to 5+}\left(2^{\frac{1}{2}} - (a-5)^{\frac{1}{2}}\right) = 2^{3/2}$$

3. Consider the recursive sequence defined by

$$a_1 = 1 \text{ and } a_{n+1} = \sqrt{6+a_n} \text{ for all } n \geq 1.$$

(a) Find a_1, a_2, a_3, and a_4. Round your answers to the nearest hundredth.

Solution: $We\ are\ given\ that\ a_1 = 1.\ \ So\ a_2 = \sqrt{6+a_1} = \sqrt{7} \approx 2.65$

Next $a_3 = \sqrt{6 + a_2} \approx \sqrt{8.65} \approx 2.94$; *Finally,* $a_4 = \sqrt{6 + a_3} \approx 2.99$

(b) Find the $\lim_{n\to\infty} a_n$ assuming that this limit exists.

Solution: Let $L = \lim_{n\to\infty} a_n$. *Then, since* $\lim a_{n+1} = \lim a_n$ *and* $\lim a_{n+1} = \sqrt{6 + \lim a_n}$, *we have:*

$L = \sqrt{6 + L}$ which implies that $L^2 - L - 6 = 0$. *Factoring*: $(L-3)(L+2) = 0$, *so* $L = -2$ *or* $L = 3$.

As L is defined to be a limit of positive numbers, we reject the possibility that L = -2.

Thus L = 3.

For each of the following *sequences*, determine convergence or divergence. In the case of divergence, find the limit of the sequence. *Show your work.*

4. $a_n = n \sin\frac{1}{n}$

Solution: Let h = 1/n. Then n = 1/h and as n → ∞, h → 0. Hence:

$$a_n = n \sin\frac{1}{n} = \frac{\sin h}{h} \to 1 \text{ as } h \to 0$$

5. $b_n = \int_0^n e^{-5x} dx$

Solution: $b_n = \int_0^n e^{-5x} dx = \left(-\frac{1}{5} e^{-5x}\Big|_0^n\right) = -\frac{1}{5}\left(e^{-5n} - 1\right) \to \frac{1}{5}$

6. $c_n = \dfrac{1}{\arctan(\ln(\ln n))}$

Solution: *Since* $\ln \ln n \to \infty$ *as* $n \to \infty$, $c_n = \dfrac{1}{\arctan(\ln(\ln n))} \to \dfrac{1}{\frac{\pi}{2}} = \dfrac{\pi}{2}$

7. $d_n = \dfrac{n(5n+1)(n-2018)^5}{1789+\ln n+(2n+3)^3(n-11)^4}$

Solution:

$$d_n = \frac{n(5n+1)(n-2018)^5}{1789+\ln n+(2n+3)^3(n-11)^4} \approx \frac{n(5n)n^5}{(2n)^3 n^4} = \frac{5}{8}$$

8. $e_n = \sqrt{n^2 + 5n} - \sqrt{n^2 - 21n}$

Solution: Rationalizing the "numerator" yields:

$$e_n = \sqrt{n^2 + 5n} - \sqrt{n^2 - 21n} = \left(\sqrt{n^2 + 5n} - \sqrt{n^2 - 21n}\right)\frac{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}} =$$

$$\frac{(n^2 + 5n) - (n^2 - 21n)}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}} = \frac{26n}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}} \to 13$$

9. $z_n = \int_2^n \frac{1}{\sqrt{x} - 1}\, dx$

Solution: Let $x = u^2$; then $dx = 2u\, du$. So

$$z_n = \int_2^n \frac{1}{\sqrt{x} - 1}\, dx = \int_{\sqrt{2}}^{\sqrt{n}} \frac{1}{u - 1}\, 2u\, du = 2\int_{\sqrt{2}}^{\sqrt{n}} \frac{u}{u - 1}\, u\, du$$

Next, let $y = u - 1$; $dy = du$. So

$$z_n = 2\int_{\sqrt{2}}^{\sqrt{n}} \frac{u}{u - 1}\, du = 2\int_{\sqrt{2}}^{\sqrt{n}} \frac{y + 1}{y}\, dy = 2\int_{\sqrt{2}}^{\sqrt{n}} \left(1 + \frac{1}{y}\right) dy =$$

$$2(y + \ln y)\Big[_{\sqrt{2}}^{\sqrt{n}} = 2\left(\sqrt{n} + \ln\sqrt{n} - \sqrt{2} - \ln\sqrt{2}\right) = \infty.$$

Hence, the sequence diverges.

10. *[2 pts each]* For each of the following statements answer ***True*** or ***False***. *Briefly justify each answer!*

(a) $x^3 \ln x + x + 1 = o(x^4)$

Answer: True

(b) $4^n = o(\pi^n)$

Answer: False

(c) $\dfrac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^5 + 5x^3 + x + 2015} = O(x^8)$

True since:

$$\frac{\dfrac{3x^3(x^2+1)^5+5x\ln x+99}{x^5+5x^3+x+2015}}{x^8} = \frac{3x^3(x^2+1)^5+5x\ln x+99}{x^{13}+5x^{11}+x^9+2015x^8} \to 3 \ \ as \ \ x \to \infty$$

(d) $(\ln x)^{2018} = o(x)$

Answer: True

(e) $\left(1+\frac{1}{x}\right)^{3x} = o(e^{4x})$

Answer: True, since $\left(1+\frac{1}{x}\right)^{3x} \to e^3$

11. Evaluate $\int \arcsin x \; dx$

Solution: Using integration by parts:

Let f(x) = arcsin x

Then $g'(x) = 1$

So g(x) = x and $f'(x) = \frac{1}{\sqrt{1-x^2}}$

Thus

$\int arcsin\, x \;\; dx = x\, arcsin\, x - \int \frac{x}{\sqrt{1-x^2}}\, dx = \; x\, arcsin\, x - \sqrt{1-x^2} + C$

12. Evaluate

$$\int x^{\frac{1}{4}} \ln x \, dx$$

Solution: Use substitution:

Let $u = x^{\frac{1}{4}}$; *then* $x = u^4$ *and so* $dx = 4\,u^3 du$

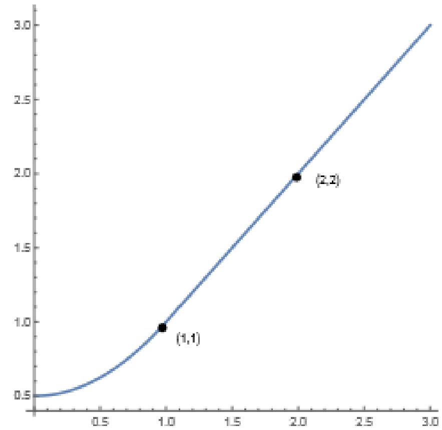
Thus

$\int x^{\frac{1}{4}}\, ln\, x\; dx = \int u\; ln(u^4)\; 4u^3 du = \int u\; 4\;(ln\, u)\; 4u^3 du = 16 \int u^4\; ln\, u\; du =$

$\frac{4}{25} x^{\frac{5}{4}}\,(5\; ln\, x - 4) + C$ *since*

$$\int u^4\; ln\, u\; du = \frac{1}{25} u^5 (5\; ln\, u - 1)$$

13. The graph of the function *p(x)* is shown below:



Use integration by parts with the selection of $f = x$ and $g' = p''(x)\,dx$ along with information from the graph to find the value of $\int_0^2 x\,p''(x)\,dx$.

Solution: Integration by parts formula

$$\int_0^2 f(x)g'(x)dx = f(x)g(x)\Big[_0^2 - \int_0^2 f'(x)g(x)dx$$

$$Now\ let\ f(x) = x\ and\ g'(x) = p''(x).$$

$$Then\ f'(x) = 1\ and\ g(x) = p'(x).$$

$$So\ \int_0^2 xp''(x)dx = f(x)g(x)\Big[_0^2 - \int_0^2 f'(x)g(x)dx = f(2)g(2) - f(0)g(0) - \int_0^2 1p'(x)dx =$$

$$2p'(2) - 0 - \left(p(x)\Big|_0^2\right) = 2p'(2) - \big(p(2) - p(0)\big) = 2p'(2) - p(2) + p(0) = 2(1) - 2 + 0.5 = \frac{1}{2}$$

Here we have estimated $p'(2)$ *using the fact that the graph is virtually linear from* $(1,1)$ *to* $(2,2)$.

For each improper integral given below, determine *convergence* or *divergence*. (You will need to use the Comparison Test here.) *Justify your answers!*

14. $\int_4^\infty \frac{1}{(\ln x\ -1)^2}dx$

Solution:

Note that $\frac{1}{(\ln x-1)^2} > \frac{1}{(\ln x)^2} > \frac{1}{x} > 0.$

Now $\int_4^\infty \frac{1}{x}dx$ *diverges; so by the Comparison Test,*

$\int_4^\infty \frac{1}{(\ln x\ -1)^2}dx$ *diverges as well.*

15. $\int_{0+}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx$

Solution:

$$\int_{0+}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx = \int_{0+}^{1} \frac{x+3}{\sqrt{x^3+x^5}} dx + \int_{1}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx$$

Now, for 0 $< x <$ *1,* $\frac{x+3}{\sqrt{x^3+x^5}} > \frac{3}{\sqrt{x^3}} = 3\, \frac{1}{x^{\frac{3}{2}}} > 0$

Now, invoking the p-test for type 2 improper integrals,

$$\int_{0+}^{1} \frac{x+3}{\sqrt{x^3+x^5}} dx \text{ diverges}$$

Hence $\int_{0+}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx$ *diverges.*

16. Evaluate the integrals below, given that f(x) is a continuous function for $0 \le x \le 6$ with the following properties:

$$f(0) = 2, \quad f(2) = 3, \quad f(4) = -1, \quad f(6) = 5; \qquad f'(0) = 1, \quad f'(2) = 4;$$
$$\int_0^2 f(x)\,dx = 3, \quad \int_2^4 f(x)\,dx = 1, \quad \int_4^6 f(x)\,dx = 6.$$

(a) $\int_0^2 x f'(x)\,dx = \mathbf{3}.$

Solution: Integration by parts.

Let u(x) = x and $v'(x) = f'(x)$. *Then* $u'(x) = 1$ *and* $v(x) = f(x)$.

$$\text{So } \int_0^2 u(x)v'(x)dx = u(x)v(x)\Big[_0^2 - \int_0^2 u'(x)v(x)dx =$$

$$2f(2) - \int_0^2 f(x)dx = 2(3) - 3 = 3$$

(b) $\displaystyle\int_2^4 f'(x)\left(2+3f(x)\right)dx \;=\; \mathbf{-20.}$

Break up the integral into two pieces. The first one is done using the FTC directly, while the second one is handled by noting $f'(x)f(x) = \frac{1}{2}\left(f^2(x)\right)'$.

$$\begin{aligned}\int_2^4 f'(x)\left(2+3f(x)\right)dx &= 2\int_2^4 f'(x)dx + \frac{3}{2}\int_2^4 \left(f^2(x)\right)'dx \\ &= 2\left(f(4)-f(2)\right) + \frac{3}{2}\left(f(4)^2 - f(2)^2\right) \;=\; -20\,.\end{aligned}$$

(c) $\displaystyle\int_0^2 f(3x)\,dx \;=\; \mathbf{\frac{10}{3}}.$

By substitution, setting $u = 3x$, *so* $dx = du/3$ *and the new limits of integration 0 and 6, we find:*

$$\int_0^2 f(3x)dx = \frac{1}{3}\int_0^6 f(u)du = \frac{1}{3}\left(\int_0^2 f(u)du + \int_2^4 f(u)du + \int_4^6 f(u)du\right) = \frac{3+1+6}{3} = \frac{10}{3}.$$

17. Consider a group of people who have received a new treatment for pneumonia. Let t be the number of days it takes for a person with pneumonia to fully recover. The probability density function giving the distribution of t is

$$f(t) = \frac{10}{(1+at)^2}, \qquad \text{for } t > 0,$$

for some positive constant a.

(a) Give a practical interpretation of the quantity $\int_3^{10}$ f(t) dt. You need not compute the value of this integral. Use complete sentences.

Solution: The fraction of the people with pneumonia who recovered during a period of three to ten days after treatment.

(b) Find a formula for the *cumulative distribution function* F(t) of f(t) for t > 0. Show all your work. Your answer may include the constant *a*. Your final answer should not include any integrals.

Solution: $\displaystyle F(t) = \int_0^t \frac{10}{(1+ax)^2}dx = -\frac{10}{a(1+ax)}\bigg|_0^t = \frac{10}{a} - \frac{10}{a(1+at)}$

(c) Determine the value of *a*. Show your work.

Solution: Since $f(t)$ is a probability density function, then $1 = \int_0^\infty \frac{10}{(1+ax)^2}dx$. Hence

$$\int_0^\infty \frac{10}{(1+ax)^2}dx = \lim_{b\to\infty}\int_0^b \frac{10}{(1+ax)^2}dx = \lim_{b\to\infty}\frac{10}{a} - \frac{10}{a(1+ab)} = \frac{10}{a}.$$

Hence $a = 10$.

18. The lifetime t (in years) of a tree has probability density function

$$f(t) = \begin{cases} \frac{a}{(t+1)^p} & \text{for} \quad t \geq 0. \\ 0 & \text{for} \quad t < 0. \end{cases}$$

where $a > 0$ and $p > 1$.

(a) Use the comparison method to find the values of p for which the average lifetime M is finite ($M < \infty$). Properly justify your answer.

Solution: The average lifetime M is given by the formula $M = \int_0^\infty t\frac{a}{(t+1)^p}dt$.
Since

$$t\frac{a}{(t+1)^p} \leq t\frac{a}{t^p} = \frac{a}{t^{p-1}} \qquad \text{for } t > 0,$$

then

$$\int_1^\infty t\frac{a}{(t+1)^p}dt \leq \int_1^\infty \frac{a}{t^{p-1}}dt$$

We know that $a\int_1^\infty \frac{1}{t^{p-1}}$ converges precisely when $p-1 > 1$ ($p > 2$) by the p-test, so the first integral converges precisely when $p > 2$. This implies that the average lifetime M is finite for $p > 2$.

(b) Find a formula for *a* in terms of *p*. Show all your work.

Solution: We know that

$$1 = \int_0^\infty \frac{a}{(t+1)^p} dt.$$

We use u-substitution with $u = t+1$ to calculate the integral:

$$\begin{aligned}\int_0^\infty \frac{a}{(t+1)^p} dt &= \lim_{b\to\infty} \int_0^b \frac{a}{(t+1)^p} dt \\ &= \lim_{b\to\infty} \int_1^{b+1} \frac{a}{u^p} du = a \lim_{b\to\infty} \int_1^{b+1} u^{-p} du \\ &= a \lim_{b\to\infty} \frac{u^{-p+1}}{(-p+1)}\Big|_1^{b+1} = a \lim_{b\to\infty} \frac{1}{(-p+1)u^{p-1}}\Big|_1^{b+1} \\ (\text{since } p > 1) \quad &= \frac{a}{p-1}.\end{aligned}$$

Therefore $1 = \frac{a}{p-1}$, so $a = p - 1$.

(c) Let C(t) be the cumulative distribution function of f(t). For a given tree, what is the practical interpretation of the expression 1 − C(30)?

Solution: 1 − C(30) is the probability that a given tree lives at least 30 years.

Extra Credit A:

[MIT integration bee]

$$\int e^{\arccos x} dx$$

Hint: Begin with a u substitution.

Solution: Let u = arccos x. Then x = cos u, and dx = -sin u.

So

$$\int e^{arccos\, x} dx = -\int e^u \sin u \, du$$

Using integration by parts (twice) we find that $\int e^u \sin u \, du = \frac{1}{2} e^u (\sin u - \cos u) + C$.

Note that cos u = x and sin u = $\sqrt{1 - \cos^2 u} = \sqrt{1 - x^2}$.

Finally,

$$\int e^u \sin u \, du = \frac{1}{2} e^u (\sin u - \cos u) + C = \frac{1}{2} e^{arccos\, x} \left(x - \sqrt{1 - x^2}\right) + C$$

Extra Credit B:

A bouncy ball is launched up 20 feet from the floor and then begins bouncing. Each time the ball bounces up from floor, it bounces up again to a height that is 60% the height of the previous bounce. (For example, when it bounces up from the floor after falling 20 ft., the ball will bounce up to a height of 0.6(20) = 12 feet.) Consider the following sequences, defined for $n \geq 1$:

- Let h_n be the height, in feet, to which the ball rises when the ball leaves the ground for the nth time. So $h_1 = 20$ and $h_2 = 12$
- Let f_n be the total distance, in feet, that the ball has traveled (both up and down) when it bounces on the ground for the nth time. For example, $f_1 = 40$ and $f_2 = 40 + 24 = 64$.

(a) Find the values of h_3 and f_3 .

Solution: $h_3 = (0.6)(12) = 7.2$

$f_3 = 64 + 14.4 = 78.4$

(b) Find a closed form expression for h_n and f_n. ("Closed form" here means that your answers should not include sigma notation or ellipses (…). Your answers should also **not involve recursive formulas**!)

Solution: $h_n = 0.6h_{n-1}$ is a recursive relationship that holds between the terms of the sequence h_n for $n > 1$, and this recursive formula means that h_n is a geometric sequence. The (constant) ratio of successive terms is equal to 0.6 and first term is $h_1 = 20$. So we see that $h_n = 20(0.6)^{n-1}$.

Note that the term f_n is twice the sum of the first n terms of the h_n sequence. (Twice because the bouncy ball travels both up and down.) We use the formula for a partial sum of a geometric series (i.e. a finite geometric series) to find

$$f_n = 2(h_1 + h_2 + ... + h_n) = 2(20 + ... + 20(0.6)^{n-1})$$
$$= \frac{2(20)(1-(0.6)^n)}{1-0.6} = \frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n).$$

Answer: $h_n =$ $20 \cdot (0.6)^{n-1}$ and $f_n =$ $\frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n)$

(c) Decide whether the given sequence converges or diverges. If it does converge, compute its sum.

(i) *The sequence $\{f_n\}$ converges to 100.*

Solution: The limit of the sequence f_n is

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{40(1-(0.6)^n)}{0.4} = \frac{40}{0.4} = 100.$$

Since this limit exists, the sequence f_n converges, and this computation shows that it converges to 100.

Alternatively, as we saw in part **b**, the sequence f_n is the sequence of partial sums of the geometric series $\sum_{k=1}^{\infty} 2h_k = \sum_{k=1}^{\infty} 40(0.6)^{k-1}$. Since $r = 0.6$ and $|0.6| < 1$, we know that this geometric series converges to $\frac{40}{1-0.6} = 100$. By definition of series convergence, this sum is the limit of the sequence of partial sums f_n, i.e. $\lim_{n\to\infty} f_n = 100$.

(ii) *The series $\sum_{n=0}^{\infty} h_n$ converges and its sum is 50.*

Solution: Next, we consider the series $\sum_{n=1}^{\infty} h_n$, which we know is geometric from part **b**. Since the common ratio between successive terms is 0.6, the series converges, and the formula for the sum of a convergent geometric series gives us

$$\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} 20 \cdot (0.6)^{n-1} = \frac{20}{1-0.6} = 50,$$

Alternatively, since the sequence f_n is the sequence of partial sums of the series $\sum_{k=1}^{\infty} 2h_k$, we have $\sum_{n=1}^{\infty} h_n = \frac{1}{2} \lim_{n\to\infty} f_n = \frac{100}{2} = 50.$

If a lion could talk, we could not understand him.

- **Ludwig Wittgenstein**