MATH 162 SOLUTIONS: TEST II 28 MARCH 2018

Instructions: Answer any 11 of the 13. You may answer more than 11 to earn extra credit.

1. Determine the value of the following two infinite series:

(a) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots$

(*Hint:* Use a well-known Maclaurin series.)

Solution:

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, replacing x by π yields:

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots = \sin \pi = 0$$

(b) $\sum_{n=1}^{\infty} \frac{13^n}{5^{2n+1}}$

Solution:

Note that this series is geometric with 1st term = $\frac{13}{5^3}$ and common ratio of $\frac{13}{25}$.

Hence

$$\sum_{n=1}^{\infty} \frac{13^n}{5^{2n+1}} = \frac{a}{1-r} = \frac{\frac{13}{5^3}}{1-\frac{13}{25}} = \frac{13}{125-65} = \mathbf{\frac{13}{60}}.$$

2. *Without* using l'Hôpital's rule, find the following limit:

$$\lim_{x \to 0} \frac{\arctan x - x}{\sin x - x}$$

Solution:

$$\frac{\arctan x - x}{\sin x - x} = \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) - x}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x} = \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots} = \frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{x^2}{5!} - \cdots} \to \frac{-\frac{1}{3}}{-\frac{1}{3!}} = 2 \; as \; x \to 0.$$

3. The ***Bessel function*** of order one is defined by its Maclaurin series, viz:

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!\,2^{2n+1}} \; x^{2n+1}$$

(a) Compute $J_1^{(2019)}(0)$. Do not simplify.

Solution:

The 2019th derivative of J_1 at x = 0 corresponds to 2019! times the 1009th coefficient in the Maclaurin series given above. (This is true since 2n+1=2019 implies that n = 1009.)

Thus $J_1^{(2019)}(0) = \dfrac{-(2019)!}{(1009)!\,(1010)!\,2^{2019}}$

(b) Find $P_5(x)$, the *Maclaurin polynomial of order 5* that approximates $J_1(x)$ near 0.

Solution:

Computing $\sum_{n=0}^{2} \frac{(-1)^n}{n!(n+1)!\,2^{2n+1}} \; x^{2n+1}$ we find:

$$P_5(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}$$

(c) Use the Maclaurin polynomial from part (b) to compute:

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3}$$

Solution:

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \lim_{x\to 0} \frac{\frac{x}{2} - \frac{1}{2}x - \frac{x^3}{16} + \frac{x^5}{384}}{x^3} = -\frac{1}{16}$$

3. Using the formula for the geometric series, find the Maclaurin series expansion of

$f(x) = \frac{1}{(1-x)^3}$. *Hint:* Differentiate basic geometric series.

What is the coefficient of x^n in this expansion?

Solution:

Begin with the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{valid for } |x| < 1.$$

Differentiate each side to obtain:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

Differentiate once again to obtain:

$$\frac{2}{(1-x)^3} = 2 + 3 \cdot 2\,x + 4 \cdot 3\,x^2 + 5 \cdot 4\,x^3 + \cdots + n(n-1)x^{n-2} + \cdots$$

Adjusting for the x^n term:

$$\frac{2}{(1-x)^3} = 2 + 3 \cdot 2\,x + 4 \cdot 3\,x^2 + 5 \cdot 4\,x^3 + \cdots + (n+2)(n+1)x^n + \cdots$$

Finally:

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(2 + 3 \cdot 2\,x + 4 \cdot 3\,x^2 + 5 \cdot 4\,x^3 + \cdots + (n+2)(n+1)x^n + \cdots \right)$$

Thus the coefficient of x^n is $\frac{(n+2)(n+1)}{2}$.

5. Given y = G(x) below, calculate the value of $G^{(1313)}(0)$. *(Express your answer in factorial form.)*

$$G(x) = x^3 \sinh(x^2)$$

Solution:

Beginning with the Maclaurin series for sinh t and then replacing t by x^2:

$$\sinh t = \frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2n+1}}{(2n+1)!} + \cdots$$

$$\sinh(x^2) = \frac{x^2}{1!} + \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots + \frac{x^{4n+2}}{(2n+1)!} + \cdots$$

Now, multiplying by x^3 yields:

$$G(x) = x^3\sinh(x^2) = \frac{x^5}{1!} + \frac{x^9}{3!} + \frac{x^{13}}{5!} + \ldots + \frac{x^{4n+5}}{(2n+1)!} + \ldots$$

Now, the general Maclaurin series of G(x) is:

$$G(x) = G(0) + \frac{G'(0)}{1!}x + \ldots + \frac{G^{(k)}(0)}{k!}x^k + \ldots$$

Thus the coefficient of x^{1313} is $G^{(1313}(0) / 1313!$

Now the coefficient of x^{1313} in the series for x^3 sinh (x^2) occurs when 4n + 5=1313, that is, when n =327 (and so 2n + 1=655). Thus this coefficient is: 1 / 655!

Equating $G^{(1313)}(0)$ / 1313! with 1 /655!, we find that:

$G^{(1313)}(0)$ = 1313! / 655!

6. For each series below, determine *absolute convergence*, *conditional convergence* or *divergence*. Justify each answer.

$$(a) \quad \sum_{n=3}^{\infty} (-1)^n (1+\frac{1}{n})^{-n^2}$$

Solution: Let $a_n = (-1)^n \left(1+\frac{1}{n}\right)^{-n^2}$

Then applying the n^{th} root test:

$$|a_n|^n = \left(\left(1+\frac{1}{n}\right)^{-n^2}\right)^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$$

Thus the series converges absolutely.

$$(b)\ \sum_{k=1}^{\infty} (-1)^k\ k\sin(1/k)$$

Solution: This series diverges by the n[th] term test: $k\sin\frac{1}{k} \to 1\ as\ k \to 0.$

7. For each of the two power series below, determine the radius of convergence. Do ***not*** investigate the behavior of each power series at the endpoints.

$$(a)\ \sum_{n=1}^{\infty} \frac{n^{13}}{13^n}\ (x-13)^n$$

Solution:

Using the ratio test:

$$\left| \frac{\dfrac{(n+1)^{13}}{13^{n+1}}(x-13)^{n+1}}{\dfrac{n^{13}}{13^n}(x-13)^n} \right| = \frac{1}{13}\frac{(n+1)^{13}}{n^{13}}|x-13| =$$

$$\frac{1}{13}(1+1/n)^{13}|x-13| \to \frac{1}{13}|x-13|$$

Thus the series converges absolutely for |x – 13| < 13.

The radius of convergence is 13.

$$(b)\ \sum_{n=1}^{\infty} \sqrt{\frac{1+n^{13}}{3+n^{14}}}\,(x-13)^n$$

Solution: *Using the ratio test:*

$$\left|\frac{\sqrt{\frac{1+(n+1)}{3+(n+1)^{14}}}\ (x-13)^{n+1}}{\sqrt{\frac{1+n^{13}}{3+n^{14}}}\ (x-13)^{n}}\right| = \sqrt{\frac{n+2}{1+n^3}}\sqrt{\frac{3+n^{14}}{3+(n+1)^{13}}}\ |x-13| \to |x-13|$$

Radius of convergence = 1

8. For the power series below, determine the interval of convergence. Investigate end-point behavior.

$$(a)\quad \sum_{n=1}^{\infty}\frac{\ln n}{n^2}(x-3)^n$$

Solution: Using the ratio test:

$$\left|\frac{\frac{(\ln(n+1)}{(n+1)^2}(x-3)^{n+1}}{\frac{\ln n}{n^2}\ (x-3)^n}\right| \to |x-3|$$

Radius of convergence = 1

When x=4, $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}(x-3)^n = \sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ *Which converges absolutely by the comparison and p-tests (since ln n <* $\sqrt{n}$ *for large n.*

When x=2, $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}(-1)^n$ *which converges absolutely from the previous analysis.*

(b)

$$\sum_{n=1}^{\infty}\frac{e^{3n}}{1+e^{4n}}x^n$$

Solution: radius of convergence = e

Using the ratio test:

$$\left|\frac{\frac{e^{3(n+1)}}{1+e^{4(n+1)}}\,x^{n+1}}{\frac{e^{3n}}{1+e^{4n}}\,x^{n}}\right| = \frac{e^{3(n+1)}}{e^{3n}}\,\frac{1+e^{4n}}{1+e^{4(n+1)}}\,|x| =$$

$$e^{3}\,\frac{1+e^{-4n}}{e^{-4n}+e^{4}}\,|x| = \frac{1}{e}\,|x|$$

Hence the radius of convergence is e.

When x = e, $\sum_{n=1}^{\infty}\frac{n^{13}}{13^{n}}\,(x-13)^{n}$

9. Find the radius of convergence of convergence:

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-4)^n$$

Solution:

Invoking the n^{th} root test:

$$\left|\left(1+\frac{1}{n}\right)^{n^2}(x-4)^n\right|^{1/n} = \left(1+\frac{1}{n}\right)^{n}|x-4| \to e|x-4|$$

Thus, the series converges absolutely for e |x – 4| < 1. So the interval of convergence is (4 – 1/e, 4 + 1/e) and the radius of convergence is 1/e.

10. For each improper integral below, determine convergence or divergence. ***Justify each answer!***

$$(A)\quad \int_{1}^{\infty}\frac{2015+\ln x}{x^{3}}\,dx$$

Solution:

Since ln x < x for x > 1,

$$0 < \frac{2015 + x}{x^3} = \frac{2016}{x^2}$$

Now using the Comparison Test, and the p-test for p = 2, we see that our improper integral converges.

$$(B) \quad \int_0^\infty \frac{1 + x^2 e^{\pi x} + (\ln x)^{2015\pi}}{\pi + e^{4x}}\, dx$$

Solution:

Observe that

$$0 < \frac{1 + x^2 e^{\pi x} + (\ln x)^{2015\pi}}{\pi + e^{4x}} < \frac{e^{\pi x} + x^2 e^{\pi x} + e^{\pi x}}{e^{4x}} =$$

$$\frac{1 + x^2 + 1}{e^{(4-\pi)x}} < \frac{3x^2}{e^{(4-\pi)x}} < \frac{3e^{\frac{(4-\pi)}{2}x}}{e^{(4-\pi)x}} = 3\frac{1}{e^{\frac{(4-\pi)}{2}x}} < 3\frac{1}{e^{\frac{(1/2)}{2}x}} = 3\frac{1}{e^{x/4}}$$

Thus, invoking the Comparison Test, our original integral converges.

11. Find the *radius of convergence* of each of the following power series:

$$(a) \quad \sum_{n=1}^{\infty} \frac{n!}{(1)(3)(5)(7)\ldots(2n-1)} x^{3n}$$

Solution:

Using the ratio test:

$$\frac{\frac{(n+1)!}{(1)(3)(5)(7)\ldots(2n-1)(2n+1)}|x|^{3n+3}}{\frac{n!}{(1)(3)(5)(7)\ldots(2n-1)}|x|^{3n}} = (n+1)\frac{1}{2n+1}|x|^3 \to \frac{1}{2}|x|^3$$

Now, the series converges absolutely when $\frac{1}{2}|x|^3 < 1$.

Thus the interval of convergence of our series is $(-\sqrt[3]{2}\ ,\ \sqrt[3]{2})$ *and the radius of convergence is* $\sqrt[3]{2}$ *.*

$$(b)\quad \sum_{n=1}^{\infty} \frac{7^n\sqrt{n^2+4}}{\left(n^{4/3}+1789\right)^3}(x-15)^n$$

Solution:

Applying the ratio test,

$$\frac{\dfrac{7^{n+1}\sqrt{(n+1)^2+4}}{\left((n+1)^{4/3}+1789\right)^3}}{\dfrac{7^n\sqrt{n^2+4}}{\left(n^{4/3}+1789\right)^3}}\frac{|x-15|^{n+1}}{|x-15|^n}=7\left(\frac{\left(n^{4/3}+1789\right)^3}{\left((n+1)^{4/3}+1789\right)^3}\right)|x-15|$$

$$=7\left(\frac{n^{4/3}+1789}{(n+1)^{4/3}+1789}\right)^3|x-15|\to 7|x-15|$$

$$\frac{\dfrac{7^{n+1}(\sqrt{(n+1)^2+4}}{\left((n+1)^{\frac{4}{3}}+1789\right)^3}}{\dfrac{7^n(\sqrt{n^2+4}}{\left(n^{\frac{4}{3}}+1789\right)^3}}|x-15|\to 7|x-15|$$

Thus the interval of convergence of our series is $\left(15-\frac{1}{7},15+\frac{1}{7}\right)$ *and the radius of convergence is* $\frac{1}{7}$ *.*

12. *Without* using l'Hôpital's rule, find:

$$\lim_{x\to 0} \frac{e^{3x^2} - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{\cos(x^4) - 1}$$

Solution:

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$$

it follows that:

$$e^{3x^2} = 1 + \frac{3x^2}{1!} + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{3!} + \frac{(3x^2)^4}{4!} + O(x^{10}) =$$

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{27x^6}{6} + \frac{81x^8}{24} + O(x^{10}) =$$

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{9x^6}{2} + \frac{27x^8}{8} + O(x^{10})$$

Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

it follows that:

$$\cos(x^4) = 1 - \frac{x^8}{2!} + O(x^{16})$$

Hence:

$$\frac{e^{3x^2} - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{\cos(x^4) - 1} =$$

$$\frac{1 + 3x^2 + \frac{9x^4}{2} + \frac{9x^6}{2} + \frac{27x^8}{8} + O(x^{10}) - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{1 - \frac{x^8}{2!} + O(x^{16}) - 1} =$$

$$\frac{\frac{27x^8}{8} + O(x^{10})}{-\frac{x^8}{2!} + O(x^{16})} \to \frac{\frac{27}{8}}{-\frac{1}{2!}} = -\frac{27}{4}$$

13. Let $f(x) = 2e^{\frac{x}{2}}$.

(a) Find $P_2(x)$, the Taylor polynomial for f(x) of degree 2 centered at x = 1.

Solution:

$$f(1) = 2\sqrt{e}; \quad f'(1) = \sqrt{e}; \quad f''(1) = \frac{1}{2}\sqrt{e}$$

So $P_2(x) = 2\sqrt{e} + \sqrt{e}\,(x-1) + \frac{1}{2}\sqrt{e}\,(x-1)^2$

(b) Graph the functions f(x) and $P_2(x)$ for $0 \le x \le 2$ on the same set of axes. Label each function clearly.

(c) Use the polynomial $P_2(x)$ that you wrote in part (a) to estimate f(0.1) and f(1.1).

TAYLOR SERIES

The Taylor series of f(x) centered at x = c is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad for\ |x| < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1$$

$$\arccos x = \frac{\pi}{2} - \arcsin x$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \text{for } |x| \le 1,\ x \ne \pm i$$

$$(x+1)^{-n} = 1 - n\,x + \frac{1}{2}\,n\,(n+1)\,x^2 - \frac{1}{6}\,n\,(n+1)\,(n+2)\,x^3 + \ldots.$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \overset{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{for all } x$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{for all } x$$

$$\tanh x = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n \left(4^n - 1\right)}{(2n)!} x^{2n-1} \qquad = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad \text{for } |x| < \frac{\pi}{2}$$