MATH 162 SOLUTIONS: TEST III

25TH APRIL 2018 (revised May 1st)

Instructions: Answer any 8 of the following 10 problems. You may answer more than 8 to earn extra credit.

1. Write each of the following in the form a + bi. *Show your work!*

(a) $3(9 - 4i) - 5(-6 - 3i)$

Answer: 57 + 3i

(b) $(3 - i)^3$

Answer: 18 – 26 i

$$(c)\ \frac{3-5i}{1+2i}$$

Answer:

$$\frac{3-5i}{1+2i} = \left(\frac{3-5i}{1+2i}\right)\frac{(1-2i)}{(1-2i)} = \frac{-7-11i}{5} = -\frac{7}{5} - \frac{11}{5}i$$

(d) $i^{1789} + i^{444} - i^{9902}$

Solution: $i^{1789} + i^{444} - i^{9902} = i^{4(447)+1} + i^{4(111)} - i^{4(247)+2} = i + 1 + 1 = 2 + i$

$(e)\ \ e^{\frac{\pi}{4}i}$

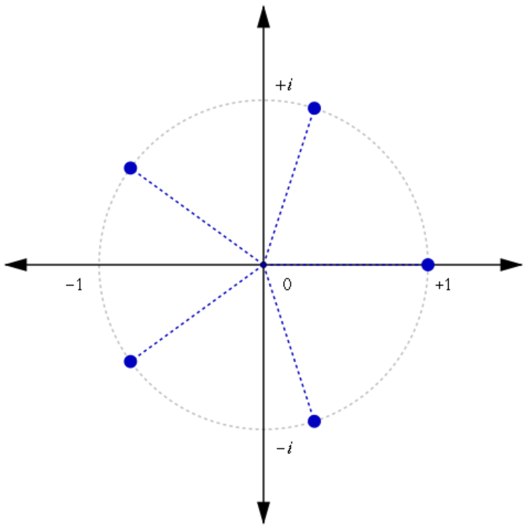
Answer: $e^{\frac{\pi}{4}i} = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\ i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\ i$

2. Find the five fifth roots of unity. That is, solve for z: $z^5 = 1$. (You may leave your answers in polar form.)

Solution: Let $z^5 = 1$. Then $z^5 = e^{0\pi i}, e^{2\pi i}, e^{4\pi i}, e^{6\pi i}, e^{8\pi i}$

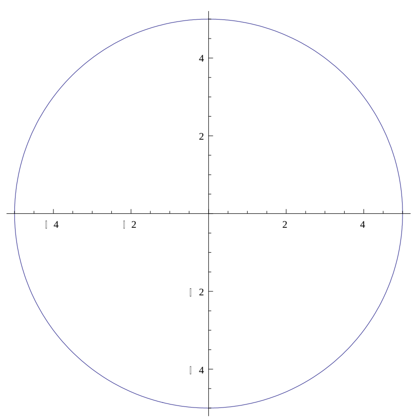
Thus $z = 1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{6\pi}{5}i}, e^{\frac{8\pi}{5}i}$

Here is a diagram of the five fifth roots of unity:



3. The base of a solid is a disk of radius 5. Each cross section cut by a plane perpendicular to a given diameter is an isosceles right triangle with hypotenuse on the base. Express the volume of the solid as a Riemann integral. You need not evaluate the integral.

Solution:



The equation of this circle is $x^2 + y^2 = 25$. *Let us assume that the diameter referred to in the question lies on the x-axis. Then, taking a typical slice at x (in the interval [-5, 5], with thickness* Δx*, the volume of the corresponding slice (an isosceles right triangle with hypotenuse* $2y = 2\sqrt{25 - x^2}$ *is given by*

$\Delta V = \frac{1}{2}\, y\,(2y)\,\Delta x = (25 - x^2)\,\Delta x$. *Thus:*

$$V = \int_{-5}^{5} (25 - x^2)\, dx$$

Using washers, we obtain (taking advantage of symmetry):

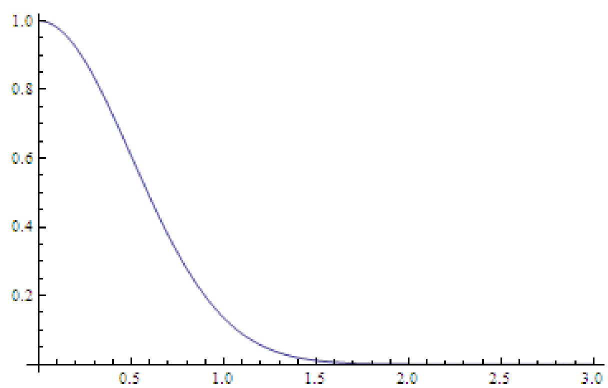
$$V = 2\int_0^3 (\pi(2x+5)^2 - \pi\, 5^2)\ dx$$

4. Let *S* be the surface of revolution obtained by rotating the curve

$$y = e^{-2x^2}, \quad 0 \le x \le 3,$$

about the line x = 9. Find a Riemann integral that expresses the surface area of this region.

(Do not evaluate the integral.)

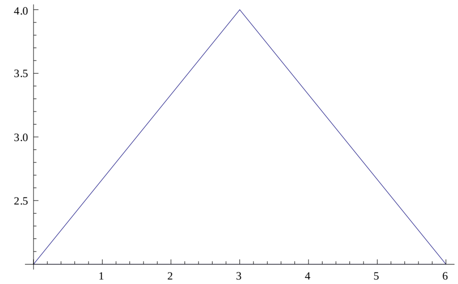


$$S = \int_{x=0}^{3} 2\pi(9-x)\, ds = \int_0^3 2\pi(9-x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx \ =$$

$$\int_0^3 2\pi(9-x)\sqrt{1+\left(-4xe^{-2x^2}\right)^2}\ \ dx = \int_0^3 2\pi(9-x)\sqrt{1+16x^2e^{-4x^2}}\ \ dx$$

5. Consider the triangle with vertices (0, 2), (6, 2), (3, 4). This triangle is rotated about the axis y = -3. Express the volume of this solid of revolution as a Riemann integral. Do not evaluate.

Solution:



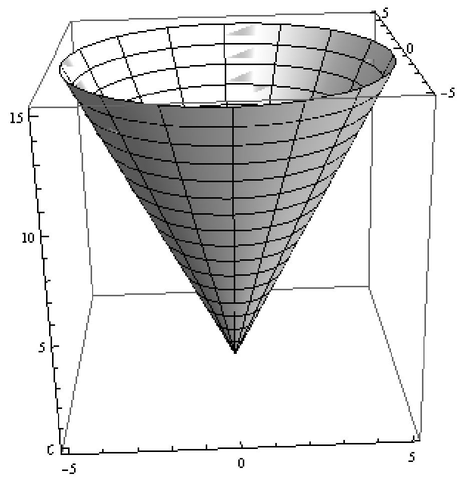
The equations of the two non-horizontal sides are: y = (2/3)x + 2 and y = (-2/3)x + 6.

Solving for x, we obtain: x = (3/2) (y – 2) and x = -(3/2) (y – 6), respectively.

Using shells, the radius of the shell at y is y – (-3) = y + 3 and the length of the shell is -(3/2) (y – 6) – ((3/2) (y – 2)) = 12 – 3y. Hence:

$$V = \int_2^6 2\pi\left(y-(-3)\right)\left(12-3y\right)dy = 6\pi\int_2^6 \left(y+3\right)\left(4-y\right)dy$$

6. A conical tank with height 15 meters and radius *5* meters is filled with a fluid of density δ kg/m^3. How much work must be done to pump all the fluid over the top rim of the tank? Do not evaluate the integral.



Solution:

First note that if y is the height of the slab of fluid that we consider and r is the radius of the top, then

$$r/y = 5/15 = 1/3$$

So r = y/3 and A(y) = $A(y) = \pi \left(\frac{y}{3}\right)^2$.

This slab must be lifted 15 – y meters. Thus the work done on the totality of slabs is:

$$W = \int_0^{15} \delta\, g \left(\frac{y}{3}\right)^2 (15 - y)\ joules$$

7. Evaluate $\int \frac{x^3+x^2+2x+1}{x^2-4x+3}\, dx$. Integrate and simplify your answer.

Solution: Performing the long division,

$$\frac{x^3 + x^2 + 2x + 1}{x^2 - 4x + 3} = x + 5 + \frac{19x - 14}{x^2 - 4x + 3}$$

Next: $\frac{19x-14}{x^2-4x+3} = \frac{A}{x-3} + \frac{B}{x-1}$

Solving: A = 43/2 and B = -5/2. Hence

$$\int \frac{x^3+x^2+2x+1}{x^2-4x+3}\, dx =$$

$$\int \left(x + 5 + \frac{43}{2}\,\frac{1}{x-3} - \frac{5}{2}\frac{1}{x-1}\right) dx =$$

$$\frac{x^2}{2} + 5x + \frac{43}{2}\ln|x-3| - \frac{5}{2}\ln|x-1| + C$$

8. Using an appropriate trigonometric substitution evaluate

$$\int \frac{\sqrt{x^2-1}}{x}\, dx$$

Solution: Let x = sec θ; then $x^2 - 1 = sec^2\theta - 1 = tan^2\,\theta$ and

$dx = \sec\theta \tan\theta\ d\theta$.

Hence

$$\int \frac{\sqrt{x^2-1}}{x}\, dx = \int \frac{\sqrt{sec^2\theta-1}}{\sec\theta} \sec\theta \tan\theta\ d\theta = \int \frac{\sqrt{sec^2\theta-1}}{\sec\theta} \sec\theta \tan\theta\ d\theta =$$

$$\int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\, d\theta = \int tan^2\theta\, d\theta = \int(sec^2\theta - 1)d\theta =$$

$$\tan\theta - \theta + C = \tan(\operatorname{arcsec} x) - \operatorname{arcsec} x + C =$$

$$\sqrt{x^2-1} - \operatorname{arcsec} x + C$$

9. A population of creatures is placed on a small preservation space. Ten creatures are initially placed on the preservation. The time it takes for such a population to reach *C* creatures is given by

$$T(C) = \int_{10}^{C} \frac{20}{x(400-x)}\, dx$$

where T is measured in years after the creatures were first placed on the preservation.

(a) Find an explicit function for T(C) by evaluating the integral given above. Be sure to show your work.

Solution: First we use partial fractions to rewrite the integrand.

$$\frac{20}{x(400-x)} = \frac{A}{x} + \frac{B}{400-x} = \frac{400A - Ax + Bx}{x(400-x)}$$

This gives us the conditions $A = B = \frac{1}{20} = 0.05$. We then have

$$\begin{aligned} T(C) &= \frac{1}{20}\int_{10}^{C}\frac{dx}{x} + \frac{1}{20}\int_{10}^{C}\frac{dx}{400-x} \\ &= \frac{1}{20}\ln|x||_{10}^{C} - \frac{1}{20}\ln|400-x||_{10}^{C} \\ &= \frac{1}{20}\ln|C| - \frac{1}{20}\ln|10| - \frac{1}{20}\ln|400-C| + \frac{1}{20}\ln|390| \\ &= \frac{1}{20}\ln|39| + \frac{1}{20}\ln\left|\frac{C}{400-C}\right| \end{aligned}$$

(b) How long does it take the creatures to reach a population of 50? State your answer in a complete sentence and include units in your answer.

Solution:

$$T(C) = \frac{1}{20}\ln|39| + \frac{1}{20}\ln\left|\frac{50}{350}\right| \approx 0.08588.$$

It takes approximately 0.08588 years (or approximately 1.0306 months) for the population of creatures to reach 50.

(c) Albertine dreamed that something terrible will happen to the creature population as C approaches 400? Can you reassure her? Explain.

Solution:

$$\begin{aligned} T(400) &= \frac{1}{20}\int_{10}^{400}\frac{dx}{x} + \lim_{b\to 400}\frac{1}{20}\int_{10}^{b}\frac{dx}{400-x} \\ &= \frac{1}{20}\ln|40| + \lim_{b\to 400}\left(-\frac{1}{20}\ln|400-b| + \frac{1}{20}\ln|390|\right) \end{aligned}$$

We know that $\lim_{b\to 400}\left(-\frac{1}{20}\ln|400-b|\right)$ diverges, so the integral diverges. This means that the time to reach 400 creatures is infinite, so the population will never reach 400 creatures.

10. Evaluate $\int \frac{1}{e^{2x}-4}\,dx$. (*Caution:* This is not a rational function.) Integrate and compute the final answer.

Solution:

$$\int \frac{1}{e^{2x}-4}\,dx = \int \frac{1}{(e^x-2)(e^x+2)}\,dx$$

Next let u = e^x. Then du = e^x dx = u dx; hence $du = \frac{1}{u}\,dx$.

$$\int \frac{1}{e^{2x}-4}\,dx = \int \frac{1}{(e^x-2)(e^x+2)}\,dx = \int \frac{1}{u(u-2)(u+2)}\,du$$

Using the technique of partial fractions:

$$\frac{1}{u(u-2)(u+2)} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{u+2}$$

So:

$$1 = A(u-2)(u+2) + Bu(u+2) + Cu(u-2)$$

Solving for A, B, C we obtain:

A = -1/4; B = 1/8; C = 1/8

Hence

$$\int \frac{1}{u(u-2)(u+2)}\,du = \int \left(\frac{\frac{1}{4}}{u} + \frac{\frac{1}{8}}{u-2} + \frac{\frac{1}{8}}{u+2} \right) du =$$

$$\frac{1}{4}\ln|u| + \frac{1}{8}\ln|u-2| + \frac{1}{8}\ln|u+2| + \mathrm{C}$$

Finally: $\int \frac{1}{e^{2x}-4}\,dx = \frac{1}{4}\ln e^{x} + \frac{1}{8}\ln|e^{x}-2| + \frac{1}{8}\ln|e^{x}+2| + C =$

$$\frac{x}{4} + \frac{1}{8}\ln|e^{x}-2| + \frac{1}{8}\ln|e^{x}+2| + C$$