DISCUSSION TOPICS & EXERCISES

22 January 2018

Part I: Rates of growth: little *o* and big *O* notation

Suppose that f(x) $\to \infty$ and g(x) $\to \infty$ as x $\to \infty$. We say that "*f is of smaller order than g* " if $\frac{f(x)}{g(x)} \to 0$ as x $\to \infty$. In this case we write f = *o*(g).

Assume that *f* and *g* are each positive for large *x*. We say that "*f is at most the order of g*" if there is a positive integer *M* for which $\frac{f(x)}{g(x)} \le M$ for large x. In this case we write f = *O*(g).

Growth rate of functions (SF University, CS Dept)

Growth-rate Functions

- O(1) – **constant** time, the time is independent of *n*, e.g. array look-up
- O(log *n*) – **logarithmic** time, usually the log is base 2, e.g. binary search
- O(*n*) – **linear** time, e.g. linear search
- O(*n**log *n*) – e.g. efficient sorting algorithms
- O(n^2) – **quadratic** time, e.g. selection sort
- O(n^k) – **polynomial** (where *k* is some constant)
- O(2^n) – **exponential** time, very slow!

- Order of growth of some common functions

  O(1) < O(log n) < O(n) < O(n * log n) < O(n^2) < O(n^3) < O(2^n)

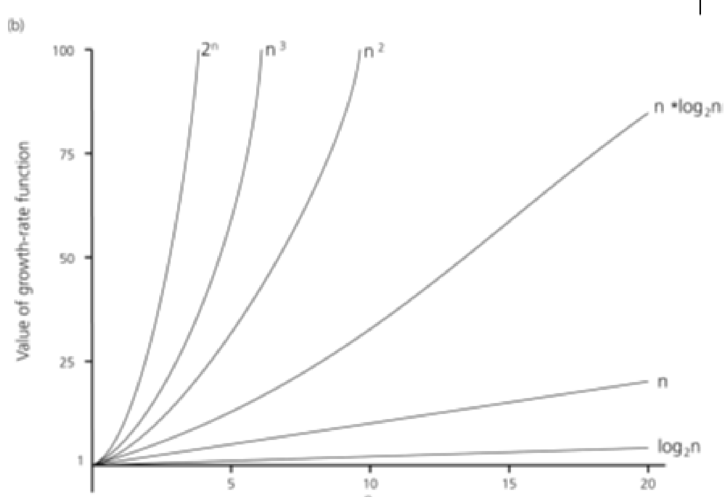
Order-of-Magnitude Analysis and Big O Notation

(a)

Function	n = 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

A comparison of growth-rate functions: a) in tabular form

Order-of-Magnitude Analysis and Big O Notation



A comparison of growth-rate functions: b) in graphical form

Arithmetic of Big-O Notation

1) If f(n) is O(g(n)) then c.f(n) is O(g(n)), where c is a constant.
   - Example: 23*log *n is* O(log *n*)

2) If $f_1(n)$ is O(g(n)) and $f_2(n)$ is O(g(n)) then also $f_1(n)+f_2(n)$ is O(g(n))
   - Example: what is order of n^2+n?
     n^2 is $O(n^2)$
     n is O(n) but also $O(n^2)$
     therefore n^2+n is $O(n^2)$

EXERCISES:

Determine which of the following statements are true; justify each answer.

(a) $3x^2 + 11 = o(x^5 + x + 99)$

(b) $x + 5 \sin x = O(x)$

(c) $2^x = o(x^{100})$

(d) $3^x = O(e^x)$

(e) $x = o(\ln x)$

(f) $3 + \ln x + \ln(\ln x) + \sqrt{x} = o\left(x^{\frac{2}{3}}\right)$

(g) $\ln x = o\left(\sqrt{x}\right)$

(h) $(x^2+1)^4 = O((2x+1)^3 x^5)$

(i) $\dfrac{x^2 + 13x + 2009}{5x + 1789} = O\left(\sqrt{x^2 + 9}\right)$

(j) $\ln x = o(\ln(\ln x))$

(k) $\ln(x^{55}+x^{33}+x^{11}+1) = O(\ln x)$

More big O examples

- 7n-2

  7n-2 is O(n)

  need c > 0 and n_0 ³ 1 such that 7n-2 £ c•n for n ³ n_0

  this is true for c = 7 and n_0 = 1

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need c > 0 and n_0 ³ 1 such that $3n^3 + 20n^2 + 5$ £ $c \cdot n^3$ for n ³ n_0

  this is true for c = 4 and n_0 = 21

- 3 log n + log log n

  3 log n + log log n is O(log n)

  need c > 0 and n_0 ³ 1 such that 3 log n + log log n £ c•log n for n ³ n_0

  this is true for c = 4 and n_0 = 2

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Exercise (Purdue University)

Arrange the following list of functions in ascending order of growth rate, i.e. if function $g(n)$ immediately follows $f(n)$ in your list then, it should be the case that $f(n) = O(g(n))$.

$$g_1(n) = 2^{\sqrt{\log n}}$$
$$g_2(n) = 2^n$$
$$g_3(n) = n^{4/3}$$
$$g_4(n) = n(\log n)^3$$
$$g_5(n) = n^{\log n}$$
$$g_6(n) = 2^{2^n}$$
$$g_7(n) = 2^{n^2}$$

Edmund Landau (1877 – 1938) is known for his work in analytic number theory and the distribution of primes. He first introduced the *little oh* notation in 1909.

PART II: REDUCTION FORMULAE

1. Find a reduction formula for $\int x^n e^x dx$

2. Find a reduction formula for $\int (\ln x)^n \, dx$

More challenging practice:

Integral	Reduction formula
$I_n = \int \frac{x^n}{\sqrt{ax+b}} \mathrm{d}x$	$I_n = \frac{2x^n\sqrt{ax+b}}{a(2n+1)} - \frac{2nb}{a(2n+1)} I_{n-1}$
$I_n = \int \frac{\mathrm{d}x}{x^n\sqrt{ax+b}}$	$I_n = -\frac{\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{a(2n-3)}{2b(n-1)} I_{n-1}$
$I_n = \int x^n\sqrt{ax+b}\,\mathrm{d}x$	$I_n = \frac{2x^n\sqrt{(ax+b)^3}}{a(2n+3)} - \frac{2nb}{a(2n+3)} I_{n-1}$