PROBLEM SET 12: DIVISIBILITY, THE GCD & LCM

$m \backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	2	6	4	10	6	14	8	18	10	22	12	26	14	30	16	34	18	38	20
3	3	6	3	12	15	6	21	24	9	30	33	12	39	42	15	48	51	18	57	60
4	4	4	12	4	20	12	28	8	36	20	44	12	52	28	60	16	68	36	76	20
5	5	10	15	20	5	30	35	40	45	10	55	60	65	70	15	80	85	90	95	20
6	6	6	6	12	30	6	42	24	18	30	66	12	78	42	30	48	102	18	114	60
7	7	14	21	28	35	42	7	56	63	70	77	84	91	14	105	112	119	126	133	140
8	8	8	24	8	40	24	56	8	72	40	88	24	104	56	120	16	136	72	152	40
9	9	18	9	36	45	18	63	72	9	90	99	36	117	126	45	144	153	18	171	180
10	10	10	30	20	10	30	70	40	90	10	110	60	130	70	30	80	170	90	190	20
11	11	22	33	44	55	66	77	88	99	110	11	132	143	154	165	176	187	198	209	220
12	12	12	12	12	60	12	84	24	36	60	132	12	156	84	60	48	204	36	228	60
13	13	26	39	52	65	78	91	104	117	130	143	156	13	182	195	208	221	234	247	260
14	14	14	42	28	70	42	14	56	126	70	154	84	182	14	210	112	238	126	266	140
15	15	30	15	60	15	30	105	120	45	30	165	60	195	210	15	240	255	90	285	60
16	16	16	48	16	80	48	112	16	144	80	176	48	208	112	240	16	272	144	304	80
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	17	306	323	340
18	18	18	18	36	90	18	126	72	18	90	198	36	234	126	90	144	306	18	342	180
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	19	380
20	20	20	60	20	20	60	140	40	180	20	220	60	260	140	60	80	340	180	380	20

Part I:

Each of the following statements is either True or False. Considering several examples, try to decide which is the case. If *False*, provide a counterexample; if *True*, try to give a complete proof. Unless otherwise indicated, all literals (a, b, c, etc.) represent *positive* integers.

1. a|a, a|0, 1|a
2. If a| (bc) then either a|b or a|c.
3. If a|(b+c) then either a|b or a|c.
4. If a|b and c|d then ac | bd.
5. If a|b and a|c then a|(b+c).
6. If a|b and a|c then a|(sb+tc).
7. If a|b then $a^3 \mid b^3$.
8. If a|b and b|a then a = b.
9. If a|b and a|c then a|(b+c)
10. If a|b and a|c then a|(bx+cy) for any integers x and y.

11. Let p be a prime. If $p^2 \mid a^2$ then $p|a$.

12. If $a|b$ and $c|b$ then $ac \mid b$.

13. If $a|b$ and $b|c$ then $a|c$.

14. Define gcd(a, b) for any integers a and b, assuming they are not both 0.

15. $\gcd(a+1, a) = 1$.

16. $\gcd(a+2, a) = 1$ or 2.

17. $\gcd(x, y) = \gcd(x - y, y)$.

18. If $a|b$ and $c|d$ then $ac \mid bd$.

19. If $a|b$ then $ac \mid bc$.

20. If $(ac) \mid (bc)$ then $a|b$.

21. If n is of the form 3K+1, then n^2 is of the form 3L+1.

22. If n is of the form 3K+2, then n^2 is of the form 3L+1.

23. Let p be a prime. If $p \mid ab$ then $p|a$ or $p|b$. (Here we must make an assumption about prime numbers.)

24. If n is of the form 5K+3, then n^2 is of the form 5L+1.

25. Consider any two consecutive natural numbers. One of the two must be even.

26. Consider any three consecutive odd natural numbers. One of the three must be divisible by 3.

27. Consider any five consecutive odd natural numbers. One of the five must be divisible by 5.

28. Let n be larger than 5. Then $n^2 - 25$ cannot be prime.

29. Let n be larger than 3. Then $n^3 - 8$ cannot be prime.

Part II:

1. State the well-ordering principle.
2. State and prove the *Division Algorithm*, assuming the divisor is positive.

3. Prove that gcd(x, y) = gcd(x – y, y).
4. Prove that, given integers a and b, not both 0, there exist integers x and y such that gcd(a, b) = ax + by.
5. Deduce from (3) that given integers *a* and *b*, not both 0, the set T = {ax + by: *x* and *y* are integers} equals the set of all multiples of the gcd(a, b).
6. Explain the significance of the following calculations:

$$
\begin{aligned}
37894060279 &= 2 \times 18272779829 + 1348500621 \\
18272779829 &= 13 \times 1348500621 + 742271756 \\
1348500621 &= 1 \times 742271756 + 606228865 \\
742271756 &= 1 \times 606228865 + 136042891 \\
606228865 &= 4 \times 136042891 + 62057301 \\
136042891 &= 2 \times 62057301 + 11928289 \\
62057301 &= 5 \times 11928289 + 2415856 \\
11928289 &= 4 \times 2415856 + 2264865 \\
2415856 &= 1 \times 2264865 + 150991 \\
2264865 &= 15 \times \underline{150991} + 0
\end{aligned}
$$

The theory of Numbers has always been regarded as one of the most obviously useless branches of Pure Mathematics. The accusation is one against which there is no valid defense; and it is never more just than when directed against the parts of the theory which are more particularly concerned with primes. A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life. The theory of prime numbers satisfies no such criteria. Those who pursue it will, if they are wise, make no attempt to justify their interest in a subject so trivial and so remote, and will console themselves with the thought that the greatest mathematicians of all ages have found it in it a mysterious attraction impossible to resist.

- G. H. Hardy from a 1915 lecture on prime numbers