PROBLEM SET 15: CARDINALITY REVISITED *(revised)*

& THE CANTOR-SCHROEDER-BERNSTEIN THEOREM

1. Show that each of the following sets has cardinality $\aleph_0$ (read Aleph-Zero).

(a) N ∪ {0, -1, -2, -3}

(b) Z, the set of integers

(c) A ∪ B where A and B are disjoint and each has cardinality $\aleph_0$.

(d) Let A_j (j = 1, 2, ..., 13) be pairwise disjoint sets each of cardinality $\aleph_0$.

Prove that $\bigcup_{j=1}^{13} A_j$ has cardinality $\aleph_0$.

(e) Let A_j (j = 1, 2, 3, ...) be a sequence of pairwise disjoint sets each of cardinality $\aleph_0$.

Prove that $\bigcup_{j=1}^{\infty} A_j$ has cardinality $\aleph_0$.

(f) A × B where each of A and B has cardinality $\aleph_0$.

(g) Prove that the set of rational numbers Q is of cardinality $\aleph_0$

2.

3. What is Cantor's ***Infinite Hotel****? Read Vilenkin's* **In Search of Infinity.**

4. Prove that the set of real numbers is *uncountable*. (Why is this called Cantor's *diagonal argument*?)

5. *True or False?* Justify each answer!

(a) The set of all complex numbers, a + bi, where a and b are integers, is countable.

(b) The set of all complex numbers, a + bi, where a and b are rational numbers, is countable.

(c) The set of all numbers of the form $x\sqrt{3} + y\sqrt[3]{5}$ is uncountable.

(d) If A is countable and $A \subseteq B$ then B is countable.

(e) If A is uncountable and $A \subseteq B$ then B is uncountable.

(f) If A is countable and B is uncountable then $A \cup B$ is uncountable.

(g) The set of all irrational numbers is uncountable.

(h) If there exists an injection $F: A \to B$, then the cardinality of A cannot be less than the cardinality of B.

(i) If there exists a surjection $G: A \to B$, then the cardinality of A cannot be less than the cardinality of B.

6. State the **Cantor-Schroeder-Bernstein Theorem**. Study its proof given below.

(from: Art of problem solving)

Schroeder-Bernstein Theorem

The **Schroeder-Bernstein Theorem** (sometimes called the **Cantor-Schroeder-Bernstein Theorem**) is a result from set theory, named for Ernst Schröder and Felix Bernstein. Informally, it implies that if two cardinalities are both less than or equal to each other, then they are equal.

More specifically, the theorem states that if A and B are sets, and there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

The proof of the theorem does not depend on the axiom of choice, but only on the classical Zermelo-Fraenkel axioms.

Proof

We call an element b of B *lonely* if there is no element $a \in A$ such that $f(a) = b$. We say that an element b_1 of B is a *descendent* of an element b_0 of B if there is a natural number n (possibly zero) such that $b_1 = (f \circ g)^n(b_0)$.

We define the function $h : A \to B$ as follows:

$$h(a) = \begin{cases} g^{-1}(a), & \text{if } f(a) \text{ is the descendent of a lonely point,} \\ f(a) & \text{otherwise.} \end{cases}$$

Note that if $f(a)$ is the descendent of a lonely point, then $f(a) = f \circ g(b)$ for some b; since g is injective, the element $g^{-1}(a)$ is well defined. Thus our function h is well defined. We claim that it is a bijection from A to B.

We first prove that h is surjective. Indeed, if $b \in B$ is the descendent of a lonely point, then $h(g(b)) = b$; and if b is not the descendent of a lonely point, then b is not lonely, so there is some $a \in A$ such that $f(a) = b$; by our definition, then, $h(a) = b$. Thus h is surjective.

Next, we prove that h is injective. We first note that for any $a \in A$, the point $h(a)$ is a descendent of a lonely point if and only if $f(a)$ is a descendent of a lonely point. Now suppose that we have two elements $a_1, a_2 \in A$ such that $h(a_1) = h(a_2)$. We consider two cases.

If $f(a_1)$ is the descendent of a lonely point, then so is $f(a_2)$. Then $g^{-1}(a_1) = h(a_1) = h(a_2) = g^{-1}(a_2)$. Since g is a well defined function, it follows that $a_1 = a_2$.

On the other hand, if $f(a_1)$ is not a descendent of a lonely point, then neither is $f(a_2)$. It follows that $f(a_1) = h(a_1) = h(a_2) = f(a_2)$. Since f is injective, $a_1 = a_2$.

Thus h is injective. Since h is surjective and injective, it is bijective, as desired. $\blacksquare$