# Problem set 16: the Cantor-Schroeder-Bernstein Theorem

 Study the proof below (from: [Art of problem solving](http://www.artofproblemsolving.com/)):



I For each problem below, first explain why *f* and *g* are injections. Then, identify the “lonely” points and (try to) identify the “descendants” of each lonely point. Finally, show how h: A B is defined.

1. Let f: N N be defined as follows: f(a) = a+3.

Let g: N N be defined as follows: g(a) = a.

(Of course, here g is a bijection. But in this elementary example. we want to illustrate how h: N N is defined.)

1. Let f: N N be defined as follows: f(a) = a+1.

Let g: N N be defined as follows: g(a) = a+1.

1. Let f: N N be defined as follows: f(a) = a+2.

 Let g: N N be defined as follows: g(a) = a+1.

 Let f: N N be defined as follows: f(a) = 2a

Let g: N N be defined as follows: g(a) = 2a+1.

1. Let f: N Z be defined as follows: f(a) = a

Let g: Z N be defined as follows: 

1. Let f: Z Z be defined as follows: f(a) = 3a

Let g: Z Z be defined as follows: g(a) = 3a+1.

## **History** (Wikipedia)

The traditional name "Schröder-Bernstein" is based on two proofs published independently in 1898. Cantor is often added because he first stated the theorem in 1895, while Schröder's name is often omitted because his proof turned out to be flawed while the name of [Richard Dedekind](https://en.wikipedia.org/wiki/Richard_Dedekind), who first proved it, is not connected with the theorem. According to Bernstein, Cantor had suggested the name *equivalence theorem* (Äquivalenzsatz).



Cantor's first statement of the theorem (1887)

* **1887** **Cantor** publishes the theorem, however without proof.
* **1887** On July 11, **Dedekind** proves the theorem (not relying on the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice)) but neither publishes his proof nor tells Cantor about it. [Ernst Zermelo](https://en.wikipedia.org/wiki/Ernst_Zermelo) discovered Dedekind's proof and in 1908 he publishes his own proof based on the *chain theory* from Dedekind's paper *Was sind und was sollen die Zahlen?*
* **1895** **Cantor** states the theorem in his first paper on set theory and transfinite numbers. He obtains it as an easy consequence of the linear order of cardinal numbers.  However, he couldn't prove the latter theorem, which is shown in 1915 to be equivalent to the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice) by [Friedrich Moritz Hartogs](https://en.wikipedia.org/wiki/Friedrich_Moritz_Hartogs).
* **1896** **Schröder** announces a proof (as a corollary of a theorem by [Jevons](https://en.wikipedia.org/wiki/William_Stanley_Jevons))
* **1896** **Schröder** publishes a proof sketch[[14]](https://en.wikipedia.org/wiki/Schr%C3%B6der%E2%80%93Bernstein_theorem%22%20%5Cl%20%22cite_note-14) which, however, is shown to be faulty by [Alwin Reinhold Korselt](https://en.wikipedia.org/wiki/Alwin_Reinhold_Korselt%22%20%5Co%20%22Alwin%20Reinhold%20Korselt) in 1911[[15]](https://en.wikipedia.org/wiki/Schr%C3%B6der%E2%80%93Bernstein_theorem#cite_note-15) (confirmed by Schröder).
* **1897** **Bernstein**, a 19 years old student in Cantor's Seminar, presents his proof.
* **1897** Almost simultaneously, but independently, **Schröder** finds a proof.
* **1897** After a visit by Bernstein, **Dedekind** independently proves the theorem a second time.
* **1898** **Bernstein'**s proof (not relying on the axiom of choice) is published by [Émile Borel](https://en.wikipedia.org/wiki/%C3%89mile_Borel%22%20%5Co%20%22%C3%89mile%20Borel) in his book on functions. (Communicated by Cantor at the 1897 [International Congress of Mathematicians](https://en.wikipedia.org/wiki/International_Congress_of_Mathematicians) in Zürich.) In the same year, the proof also appears in **Bernstein'**s dissertation.

Both proofs of Dedekind are based on his famous memoir *Was sind und was sollen die Zahlen?* and derive it as a corollary of a proposition equivalent to statement C in Cantor's paper which reads *A* ⊆ *B* ⊆ *C* and |*A*|=|*C*| implies |*A*|=|*B*|=|*C*|. Cantor observed this property as early as 1882/83 during his studies in set theory and transfinite numbers and therefore (implicitly) relying on the [Axiom of Choice](https://en.wikipedia.org/wiki/Axiom_of_Choice).

