PROBLEM SET 16: THE CANTOR-SCHROEDER-BERNSTEIN THEOREM

Study the proof below (from: Art of problem solving):

Schroeder-Bernstein Theorem

The **Schroeder-Bernstein Theorem** (sometimes called the **Cantor-Schroeder-Bernstein Theorem**) is a result from set theory, named for Ernst Schröder and Felix Bernstein. Informally, it implies that if two cardinalities are both less than or equal to each other, then they are equal.

More specifically, the theorem states that if A and B are sets, and there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

The proof of the theorem does not depend on the axiom of choice, but only on the classical Zermelo-Fraenkel axioms.

Proof

We call an element b of B *lonely* if there is no element $a \in A$ such that $f(a) = b$. We say that an element b_1 of B is a *descendent* of an element b_0 of B if there is a natural number n (possibly zero) such that $b_1 = (f \circ g)^n(b_0)$.

We define the function $h : A \to B$ as follows:

$$h(a) = \begin{cases} g^{-1}(a), & \text{if } f(a) \text{ is the descendent of a lonely point,} \\ f(a) & \text{otherwise.} \end{cases}$$

Note that if $f(a)$ is the descendent of a lonely point, then $f(a) = f \circ g(b)$ for some b; since g is injective, the element $g^{-1}(a)$ is well defined. Thus our function h is well defined. We claim that it is a bijection from A to B.

We first prove that h is surjective. Indeed, if $b \in B$ is the descendent of a lonely point, then $h(g(b)) = b$; and if b is not the descendent of a lonely point, then b is not lonely, so there is some $a \in A$ such that $f(a) = b$; by our definition, then, $h(a) = b$. Thus h is surjective.

Next, we prove that h is injective. We first note that for any $a \in A$, the point $h(a)$ is a descendent of a lonely point if and only if $f(a)$ is a descendent of a lonely point. Now suppose that we have two elements $a_1, a_2 \in A$ such that $h(a_1) = h(a_2)$. We consider two cases.

If $f(a_1)$ is the descendent of a lonely point, then so is $f(a_2)$. Then $g^{-1}(a_1) = h(a_1) = h(a_2) = g^{-1}(a_2)$. Since g is a well defined function, it follows that $a_1 = a_2$.

On the other hand, if $f(a_1)$ is not a descendent of a lonely point, then neither is $f(a_2)$. It follows that $f(a_1) = h(a_1) = h(a_2) = f(a_2)$. Since f is injective, $a_1 = a_2$.

Thus h is injective. Since h is surjective and injective, it is bijective, as desired. ■

I For each problem below, first explain why *f* and *g* are injections. Then, identify the "lonely" points and (try to) identify the "descendants" of each lonely point. Finally, show how h: A → B is defined.

(A) Let f: N → N be defined as follows: f(a) = a+3.

Let g: N → N be defined as follows: g(a) = a.

(Of course, here g is a bijection. But in this elementary example. we want to illustrate how h: N → N is defined.)

(B) Let f: N → N be defined as follows: f(a) = a+1.

Let g: N → N be defined as follows: g(a) = a+1.

(C) Let f: N → N be defined as follows: f(a) = a+2.

Let g: N → N be defined as follows: g(a) = a+1.

Let f: N → N be defined as follows: f(a) = 2a

Let g: N → N be defined as follows: g(a) = 2a+1.

(D) Let f: N → Z be defined as follows: f(a) = a

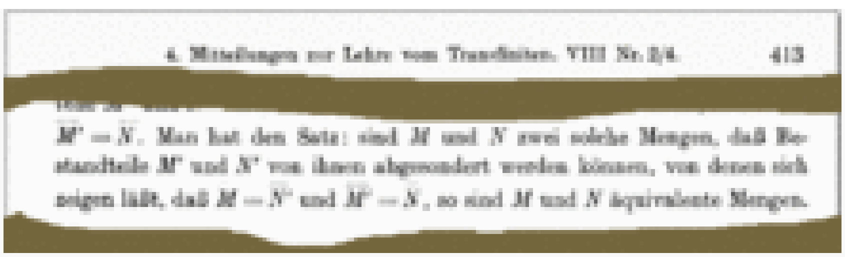
Let g: Z → N be defined as follows: $g(a) = \begin{cases} 2a+2 & if \;\; a \geq 0 \\ -2a+3 & if \;\; a \leq -1 \end{cases}$

(E) Let f: Z → Z be defined as follows: f(a) = 3a

Let g: Z → Z be defined as follows: g(a) = 3a+1.

History (Wikipedia)

The traditional name "Schröder-Bernstein" is based on two proofs published independently in 1898. Cantor is often added because he first stated the theorem in 1895, while Schröder's name is often omitted because his proof turned out to be flawed while the name of Richard Dedekind, who first proved it, is not connected with the theorem. According to Bernstein, Cantor had suggested the name *equivalence theorem* (Äquivalenzsatz).


4. Mitteilungen zur Lehre vom Transfiniten. VIII Nr. 2/4. 413

$M' \sim N$. Man hat den Satz: sind M und N zwei solche Mengen, daß Bestandteile M' und N' von ihnen abgesondert werden können, von denen sich zeigen läßt, daß $M \sim N'$ und $M' \sim N$, so sind M und N äquivalente Mengen.

Cantor's first statement of the theorem (1887)

- **1887 Cantor** publishes the theorem, however without proof.
- **1887** On July 11, **Dedekind** proves the theorem (not relying on the axiom of choice) but neither publishes his proof nor tells Cantor about it. Ernst Zermelo discovered Dedekind's proof and in 1908 he publishes his own proof based on the *chain theory* from Dedekind's paper *Was sind und was sollen die Zahlen?*
- **1895 Cantor** states the theorem in his first paper on set theory and transfinite numbers. He obtains it as an easy consequence of the linear order of cardinal numbers. However, he couldn't

prove the latter theorem, which is shown in 1915 to be equivalent to the axiom of choice by Friedrich Moritz Hartogs.

- **1896 Schröder** announces a proof (as a corollary of a theorem by Jevons)
- **1896 Schröder** publishes a proof sketch[14] which, however, is shown to be faulty by Alwin Reinhold Korselt in 1911[15] (confirmed by Schröder).
- **1897 Bernstein**, a 19 years old student in Cantor's Seminar, presents his proof.
- **1897** Almost simultaneously, but independently, **Schröder** finds a proof.
- **1897** After a visit by Bernstein, **Dedekind** independently proves the theorem a second time.
- **1898 Bernstein'**s proof (not relying on the axiom of choice) is published by Émile Borel in his book on functions. (Communicated by Cantor at the 1897 International Congress of Mathematicians in Zürich.) In the same year, the proof also appears in **Bernstein'**s dissertation.

Both proofs of Dedekind are based on his famous memoir *Was sind und was sollen die Zahlen?* and derive it as a corollary of a proposition equivalent to statement C in Cantor's paper which reads $A \subseteq B \subseteq C$ and $|A|=|C|$ implies $|A|=|B|=|C|$. Cantor observed this property as early as 1882/83 during his studies in set theory and transfinite numbers and therefore (implicitly) relying on the Axiom of Choice.