Math 201: problem set VI

Isomorphism

1. Let X be a set endowed with operation ★ and let Y be a set endowed with operation ♣. (It is assumed, of course, that each set is closed under its own operation).
   (Sometimes X and Y will have multiple operations.)

   An isomorphism is a bijection $\Phi : X \to Y$ that preserves operations.
   More precisely: For all x_1 and $x_2 \in X$, $\Phi(x_1 ★ x_2) = \Phi(x_1) ♣ \Phi(x_2)$.
   For example, let X be the space of 2×1 (column) vectors with real entries, endowed with the usual matrix addition. Let Y be the space of 1×2 (row) vectors, also endowed with the usual matrix addition.
   Define $\Phi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = [a \quad b]$

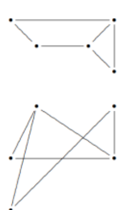
   Prove that Φ is an isomorphism.

2. Consider the following function $\Phi: \mathbb{Z} \to \mathbb{Z}$ defined by the rule, for all $a \in \mathbb{Z}$, $\Phi(a) = a + 1$.
   Assume that operation on Z is addition in both the domain and target.
   (1) Is Φ well-defined?
   (2) Is Φ surjective?
   (3) Is Φ injective?
   (4) Is Φ bijective?
   (5) Is Φ an isomorphism?

   Investigate each of the following functions by going through the list of five questions in #2.

3. Let X be the space of all polynomials with real coefficents.
   Let X and Y have one operation: polynomial addition.
   Define $F: X \to Y$ by the following rule: for all $p \in X$, $F(p) = F'$

4. Are the following two graphs isomorphic?

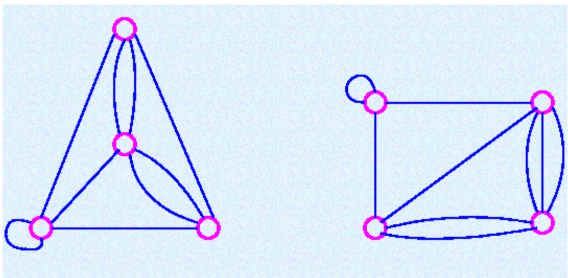


5. Let X be the set of polynomials of degree 2 or smaller with real coefficients, endowed with the usual addition of polynomials. Let Y be the set of 3×1 column vectors, endowed with the usual matrix addition. Define $G: X \to Y$ as follows: For all $a, b, c \in R$ $G(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

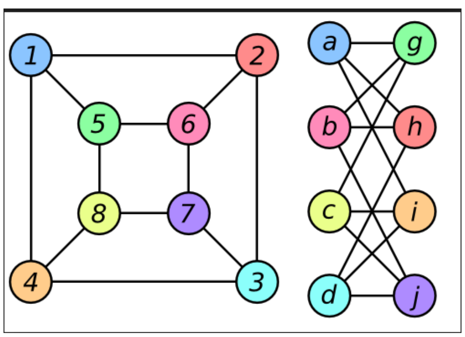
6. Let X and Y each be the set of complex numbers endowed with the usual addition and multiplication. Note, here each set has two operations.
Define H: X → Y as follows: For all z∈X H(z) = conjugate of z.

7. Let X be the set of real numbers endowed with the usual addition. Let Y be the set of all non-negative real numbers endowed with the usual multiplication.
Define T: X → Y as follows: For all s∈X T(s) = 2^s.

8. Are the following two graphs isomorphic?



11. Are the following two graphs isomorphic?



The origins of graph theory are humble, even frivolous.

- N. Biggs, E. K. Lloyd, and R. J. Wilson (*Graph Theory: 1736-1937)*