Math 201: problem set VII

Mathematical Induction

**I** 1.What is the flaw in the “induction proof” that: given *n* spiders, if one is a tarantula, then all the spiders are tarantulas?

2.Review the argument that for all natural numbers *n*, any 2n  2n chessboard with one square removed can be tiled by tri-ominos. Try to explain it to another member of your group!

3. Prove independently (of question #2) that 3 is a divisor of 22n – 1.

**II** Using the method of mathematical induction, verify each of the following:

1. 3 | (n3 + 2n) for all natural numbers *n*. (recall the notation: a|b means that b is a multiple of a)
2. 5 | (7n – 2n) for all natural numbers *n*.
3. 1 + 3 + 5 + … + (2n – 1) = n2 for all natural numbers *n*.
4. (1 + x)n ≥ 1 + nx for all real x ≥ 0 and all natural numbers *n*. (This is called *Bernoulli’s inequality*.)
5. 1 + 2 + 3 + … + n = n(n+1)/2 for all natural numbers *n*.
6. 12 + 22 + 32 + … + n2 = n(n+1)(2n+1)/6 for all natural numbers *n*.
7. 2 + 22 + 23 + … + 2n = 2n+1 – 2 for all natural numbers *n*.
8. 4n < 2n for all natural numbers n ≥ 5.
9. (1)(2) + (2)(3) + (3)(4) + … + (n)(n+1) = n(n+1)(n+2)/3 for all natural numbers *n*.
10. 2n > n2 for all natural numbers n ≥ 5.
11.  for all natural numbers *n*.
12. Consider the sequence {an} of Fibonacci numbers defined recursively by

a1 = a2 = 1, and

an+2 = an + an+1 for all n ≥ 1.

Prove that gcd (an, an+1) = 1 for all natural numbers *n*.

1. 133 | (122n – 11n) for all non-negative integers *n*.
2. (d/dx) xn = n xn-1 for all natural numbers *n*. (You may use the product rule.)
3. Prove that, for all natural numbers *n*, any 2n  2n chessboard with one square removed can be tiled by tri-ominos. Prove independently that 3 is a divisor of 22n – 1

**III** State the Principle of *Strong Induction*.

1. Using strong induction prove that every integer n ≥ 2 can be expressed as a product of primes.

2. Consider the Lucas series 1, 3, 4, 7, 11, 18, 29, 47, 76, …. This sequence is defined recursively by: a1 = 1, a2 = 3, and, for all n ≥ 3, an = an-1 + an-2. Using strong induction prove that an < (7/4)n for all positive integers *n*.

3. Define a sequence recursively by: b1 = 1, b2 = 2, b3 = 3, and, for all n ≥ 4, bn = bn-1 + bn-2 + bn-3. Using strong induction, prove that bn < 2n for all positive integers, *n*.

4. (Putnam Exam, 2007, A1) Show that every positive integer is a sum of one or more numbers of the form 2r3s, where *r* and *s* are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

*Induction makes you feel guilty for getting something out of nothing, and it is artificial, but it is one of the greatest ideas of civilization.*

- [Herbert Wilf](http://www.math.upenn.edu/~wilf/)