Math 201: problem set VII

Mathematical Induction

I

1. What is the flaw in the "induction proof" that: given *n* spiders, if one is a tarantula, then all the spiders are tarantulas?

2. Review the argument that for all natural numbers *n*, any $2^n \times 2^n$ chessboard with one square removed can be tiled by tri-ominos. Try to explain it to another member of your group!

3. Prove independently (of question #2) that 3 is a divisor of $2^{2n} - 1$.

II Using the method of mathematical induction, verify each of the following:

1. $3 \mid (n^3 + 2n)$ for all natural numbers *n*. (recall the notation: $a|b$ means that b is a multiple of a)
2. $5 \mid (7^n - 2^n)$ for all natural numbers *n*.
3. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all natural numbers *n*.
4. $(1 + x)^n \geq 1 + nx$ for all real $x \geq 0$ and all natural numbers *n*. (This is called *Bernoulli's inequality*.)
5. $1 + 2 + 3 + \ldots + n = n(n+1)/2$ for all natural numbers *n*.
6. $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for all natural numbers *n*.
7. $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$ for all natural numbers *n*.
8. $4n < 2^n$ for all natural numbers $n \geq 5$.
9. $(1)(2) + (2)(3) + (3)(4) + \ldots + (n)(n+1) = n(n+1)(n+2)/3$ for all natural numbers *n*.
10. $2^n > n^2$ for all natural numbers $n \geq 5$.
11. $13 \mid \left(8^{2^n} - 5^{2^n}\right)$ for all natural numbers *n*.
12. Consider the sequence $\{a_n\}$ of Fibonacci numbers defined recursively by

$$a_1 = a_2 = 1, \text{ and}$$

$$a_{n+2} = a_n + a_{n+1} \text{ for all } n \geq 1.$$

Prove that gcd (a_n, a_{n+1}) = 1 for all natural numbers *n*.

13. 133 | ($12^{2n} - 11^n$) for all non-negative integers *n*.

14. $(d/dx)\, x^n = n\, x^{n-1}$ for all natural numbers *n*. (You may use the product rule.)

15. Prove that, for all natural numbers *n*, any $2^n \times 2^n$ chessboard with one square removed can be tiled by tri-ominos. Prove independently that 3 is a divisor of $2^{2n} - 1$

III State the Principle of *Strong Induction*.

1. Using strong induction prove that every integer $n \geq 2$ can be expressed as a product of primes.

2. Consider the Lucas series 1, 3, 4, 7, 11, 18, 29, 47, 76, …. This sequence is defined recursively by: $a_1 = 1$, $a_2 = 3$, and, for all $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. Using strong induction prove that $a_n < (7/4)^n$ for all positive integers *n*.

3. Define a sequence recursively by: $b_1 = 1$, $b_2 = 2$, $b_3 = 3$, and, for all $n \geq 4$, $b_n = b_{n-1} + b_{n-2} + b_{n-3}$. Using strong induction, prove that $b_n < 2^n$ for all positive integers, *n*.

4. (Putnam Exam, 2007, A1) Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where *r* and *s* are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

Induction makes you feel guilty for getting something out of nothing, and it is artificial, but it is one of the greatest ideas of civilization.

- Herbert Wilf