## **PROBLEM SET 9: REVIEW QUESTIONS**

1. *Critique* the proof below: Proof is correct as is? Error in algebra or arithmetic? Stylistic errors? Logical errors?

Proposition: ∀n ∈ N,  $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$ Proof: For any n, let S<sub>n</sub> be the statement  $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$ . Base Case: S<sub>1</sub> is true, since  $1^2 = 1(1 + 1)(2 + 1)/6$ 

## **Inductive Step:**

Assume that  $S_k$  is true for  $\forall k \in N$ . Then  $1^2 + 2^2 + 3^2 + ... + (k+1)^2 = (k+1)(k+1+1))(2(k+1)+1)/6$   $k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)(k+1+1))(2(k+1)+1)/6$ Factoring:  $(1/6)(k+1)\{k(2k+1) + 6(k+1) = (k+1)(k+2)(2k+3)/6$   $(1/6) (k+1)(2k^2 + 7k + 6) = (k+1)(k+2)(2k+3)/6$ Factoring the left-hand side: (k+1)(k+2)(2k+3)/6 = (k+1)(k+2)(2k+3)/6This is a true statement! Thus we have shown that  $S_k$  is always true.

## 2. Evaluate the proof of the following statement:

**Proposition**: For all  $n \in N$ ,  $4|(3^{2n} + 7)$ .

Proof: For  $n \in N$  let  $S_n$  be the statement  $4|(3^{2n} + 7)$ .

**Base Case:** S<sub>1</sub> is true, since  $3^2 + 7 = 16$  is divisible by 4.

## **Inductive Step:**

For a given  $k \ge 1$ , assume that  $S_k$  is true, so that  $4|(3^{2k} + 7)$ . Then  $3^{2k} + 7 = 4L$  for some  $L \in \mathbb{Z}$ .

Now, 
$$3^{2(k+1)} + 7 = 9(3^{2k}) + 7 = 8(3^{2k}) + 3^{2k} + 7 = 8(3^{2k}) + 4L = 4(2(3^{2k}) + L).$$

So  $4|(32^{(k+1)} + 7)$ , and we see that  $S_{k+1}$  is true.

Hence  $S_k$  implies  $S_{k+1}$ .

Therefore, by the principle of mathematical induction,  $S_n$  is true for all  $n \in N$ .

 $\rightarrow$  Which of the following statements is correct?

a) The proposition is false but the proof is correct.

b) The proof contains arithmetic mistakes which make it incorrect.

c) The proof incorrectly assumes what it is trying to prove.

d) The proof is a correct proof of the stated result.

e) None of the above.

3. (a) Show that there is no integer x satisfying the equation 2x + 1 = 5x - 4(Hint: evaluate mod 3 and see what happens)

(b) Show that there is no integer x satisfying the equation  $18x^2 + 39x - 7 = 0$ 

(c) Show that the system of equations

$$11x - 5y = 7$$
  
9x + 10y = -3  
(Hint: reduce mod 5)

has

(d) Show that the system of equations

$$24x - 5y = 10, 11x - 9y = 13.$$

has no integer solutions.

4. Let X, Y, Z be non-empty sets. Let  $f: X \to Y$  be surjective and let g:  $Y \rightarrow Z$  be surjective. Must it follow that the composition of the two functions  $g \circ f : X \rightarrow Z$  be surjective? *Give proof or counterexample.* 

5. Is the converse to problem # 2 true? *Give proof or counterexample.* (Begin by stating the converse!)

6. (a) Is  $4^{100}$  divisible by 3? (use modular arithmetic)

(b) What is the last digit in the expansion of  $4^{100}$ ?

7. (a) Using *mathematical induction*, prove that  $10^{n+1} + 4(10^n) + 4$  is divisible by 9, for all positive integers n.

(b) Prove the result of part (a) by using *modular arithmetic*! Which is easier?

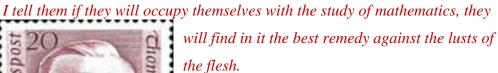
8. Let  $A = \{1, 2, 3, 4, 5\}$ , and let

 $R=\{(1,\,1),\,(1,\,3),\,(1,\,4),\,(2,\,2),\,(2,\,5),\,(3,\,1),\,(3,\,3),\,(3,\,4),\,(4,\,1),\,(4,\,3),\,(4,\,4),\,(5,\,2),\,(4,\,3),\,(4,\,4),\,(5,\,2),\,(4,\,3),\,(4,\,4),\,(5,\,2),\,(4,\,3),\,(4,\,4),\,(5,\,2),\,(4,\,3),\,(4,\,3),\,(4,\,4),\,(5,\,2),\,(4,\,3),\,(4,\,3),\,(4,\,4),\,(5,\,2),\,(4,\,3),\,($ 

(5, 5)} define an equivalence relation on A. (You may wish to check this!)

Which of the following is an *equivalence class*?

a)  $\{1, 2, 3\}$  b)  $\{2, 3, 5\}$  c)  $\{1, 3, 4\}$  d)  $\{1, 2\}$  e)  $\{1, 2, 3, 4, 5\}$ 



- Thomas Mann, The Magic Mountain



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