PROBLEM SET 9: REVIEW QUESTIONS

1. *Critique* the proof below: Proof is correct as is? Error in algebra or arithmetic? Stylistic errors? Logical errors?

Proposition: $\forall n \in N$, $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$

Proof: For any n, let S_n be the statement $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$.

Base Case: S_1 is true, since $1^2 = 1(1 + 1)(2 + 1)/6$

Inductive Step:

Assume that S_k is true for $\forall k \in N$.

Then $1^2 + 2^2 + 3^2 + ... + (k+1)^2 = (k+1)(k+1+1))(2(k+1)+1)/6$

$k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)(k+1+1))(2(k+1)+1)/6$

Factoring: $(1/6)(k+1)\{k(2k+1) + 6(k+1) = (k+1)(k+2)(2k+3)/6$

$(1/6) (k+1)(2k^2 + 7k + 6) = (k+1)(k+2)(2k+3)/6$

Factoring the left-hand side:

$(k+1)(k+2)(2k+3)/6 = (k+1)(k+2)(2k+3)/6$

This is a true statement!

Thus we have shown that S_k is always true.

2. *Evaluate the proof of the following statement:*

Proposition: For all $n \in N$, $4|(3^{2n} + 7)$.

Proof: For $n \in N$ let S_n be the statement $4|(3^{2n} + 7)$.

Base Case: S_1 is true, since $3^2 + 7 = 16$ is divisible by 4.

Inductive Step:

For a given $k \geq 1$, assume that S_k is true, so that $4|(3^{2k} + 7)$. Then $3^{2k} + 7 = 4L$ for some $L \in Z$.

Now, $3^{2(k+1)} + 7 = 9(3^{2k}) + 7 = 8(3^{2k}) + 3^{2k} + 7 = 8(3^{2k}) + 4L = 4(2(3^{2k}) + L)$.

So $4|(32^{(k+1)} + 7)$, and we see that S_{k+1} is true.

Hence S_k implies S_{k+1}.

Therefore, by the principle of mathematical induction, S_n is true for all $n \in N$.

→ Which of the following statements is correct?

a) The proposition is false but the proof is correct.

b) The proof contains arithmetic mistakes which make it incorrect.

c) The proof incorrectly assumes what it is trying to prove.

d) The proof is a correct proof of the stated result.

e) None of the above.

3. (a) Show that there is no integer x satisfying the equation $2x + 1 = 5x - 4$
(Hint: evaluate mod 3 and see what happens)

(b) Show that there is no integer x satisfying the equation $18x^2 + 39x - 7 = 0$

(c) Show that the system of equations

$$11x - 5y = 7$$
$$9x + 10y = -3$$

has no integer solution. (Hint: reduce mod 5)

(d) Show that the system of equations

$$24x - 5y = 10,$$
$$11x - 9y = 13.$$

has no integer solutions.

4. Let X, Y, Z be non-empty sets. Let $f: X \to Y$ be surjective and let $g: Y \to Z$ be surjective. Must it follow that the composition of the two functions $g \circ f : X \to Z$ be surjective? *Give proof or counterexample.*

5. Is the converse to problem # 2 true? *Give proof or counterexample.*
(Begin by stating the converse!)

6. (a) Is 4^{100} divisible by 3? (use modular arithmetic)

(b) What is the last digit in the expansion of 4^{100} ?

7. (a) Using *mathematical induction*, prove that $10^{n+1} + 4(10^n) + 4$ is divisible by 9, for all positive integers n.

(b) Prove the result of part (a) by using *modular arithmetic*!

Which is easier?

8. Let A = {1, 2, 3, 4, 5}, and let

R = {(1, 1), (1, 3), (1, 4), (2, 2), (2, 5), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 2), (5, 5)} define an equivalence relation on A. (You may wish to check this!)

Which of the following is an *equivalence class*?

a) {1, 2, 3} b) {2, 3, 5} c) {1, 3, 4} d) {1, 2} e) {1, 2, 3, 4, 5}

I tell them if they will occupy themselves with the study of mathematics, they

will find in it the best remedy against the lusts of the flesh.

- Thomas Mann, **The Magic Mountain**