MATH 201 solutions: QUIZ II Take-h0me

1. Prove that an *acyclic* connected graph with at least two vertices must have a leaf.

*Solution: Here is one of several ways of proving this result.*

*Assume that G is an acyclic connected graph with N vertices, where N > 1.*

*Choose any vertex; call it V1. Then choose an edge, V1V2, where V2 is a vertex not visited already. (This is possible since N >1).*

*Next choose an edge, V2V3, where V3 has not been visited earlier. If there is no such vertex, then the degree of V2 is 1.*

*Next, choose an edge, V3V4, where V4 has not been visited earlier. If there is no such vertex, then either the degree of V3 is 1 or we have found a cycle. The latter is not possible since G is acyclic.*

*We continue to choose vertices in this manner (one says, “inductively”), until we find a vertex, VK, of degree 1. Since there are but N vertices, this process must halt before K = N+1. When it halts, we will have found a vertex of degree 1 (or a cycle, which is not possible).*

1. How many trailing zeroes are there in the expansion of 100! (A computer solution will earn no credit.) For example, 10! has two trailing zeroes. (*Extra credit:* Same question for 1000!)

*Solution: Since the only way that a trailing 0 can appear is to have factors 2 and 5. Clearly the number of factors 2 is at least as great as the number of 5s. So we must compute the highest power of 5 that is a factor of 100!*

*This amounts to considering the multiples of 5 that are no bigger than 100, namely: 5, 10, 15, 20, 25, 30, ..., 100.*

*Now each member of this sequence, with the exception of 25, 50, 75 and 100, (which are divisible by 52) contributes one factor of 5. Also each of 25, 50, 75 and 100 contributes two factors of five.*

*Hence the number of trailing zeros of 100! is 20 + 4 = 24.*

1. Prove by mathematical induction that n! < nn for n 2.

*Solution: For* n 2 l*et Hn be the statement* n! < nn*.*

***Base case:*** *Consider H 2 : 2! < 22. LHS = 2! = 2; RHS = 22 = 4. Since LHS < RHS, the base case, H 2 , is true.*

***Inductive step:*** *Let k be a* **given** *integer that is greater than or equal to 2.*

*We assume that H k is true, namely: k! < kk.*

*Since k + 1 > 0: (k+1)! < (k+1) kk < (k + 1) (k+1)k = (k+1)k+1 and so H k+1  is true.*

1. Let X, Y, Z be non-empty sets. Let f: X → Y be injective and let g: Y → Z be injective. Must it follow that the composition of the two functions

g∘f : X →Z be injective? Recall the definition of composition, *viz.*

x X g∘f (x) = g(f(x)).

*Give proof or counterexample.*

*Solution: Yes, it must follow that* g∘f : X →Z is injective.

*Proof: Assume that there exist a, b X such that* g∘f *(a) =* g∘f *(b).*

*That is, g(f(a)) = g(f(b)). Now since g is injective f(a) = f(b).*

*Since g is injective, it follows that a = b.*

*Thus* g∘f (x) *is injective.*

1. Is the converse to problem # 4 true? *Give proof or counterexample.*

(Begin by stating the converse!)

*Solution:*

*The converse states:*

*Let X, Y, Z be non-empty sets. Let f: X → Y and g: Y → Z.*

*Assume that g∘ f : X → Z is injective. Then it follows that both f and g are injective.*

*The converse is false; here is a* ***counterexample****:*

*Let X = {1, 2}, Y = {1, 2, 3} and Z = {1, 2}.*

*Define f:* X → Y *as follows: f(1) = 1 and f(2) = 2.*

*Define* g: Y → Z *as follows: g(1) = 1, g(2) = 2 and g(3) = 1.*

*Notice that g is not injective since g(1) = g(3).*

*However g∘ f : X → Z is injective since:*

*g∘ f(1) = 1*

*g∘ f(2) = 2*

*Induction makes you feel guilty for getting something out of nothing, and it is artificial, but it is one of the greatest ideas of civilization.*

- [Herbert Wilf](http://www.math.upenn.edu/~wilf/)