MATH 201 SOLUTIONS: QUIZ II TAKE-HOME

1. Prove that an *acyclic* connected graph with at least two vertices must have a leaf.

Solution: Here is one of several ways of proving this result.

Assume that G is an acyclic connected graph with N vertices, where N > 1.

Choose any vertex; call it V_1. Then choose an edge, V_1V_2, where V_2 is a vertex not visited already. (This is possible since N >1).

Next choose an edge, V_2V_3, where V_3 has not been visited earlier. If there is no such vertex, then the degree of V_2 is 1.

Next, choose an edge, V_3V_4, where V_4 has not been visited earlier. If there is no such vertex, then either the degree of V_3 is 1 or we have found a cycle. The latter is not possible since G is acyclic.

We continue to choose vertices in this manner (one says, "inductively"), until we find a vertex, V_K, of degree 1. Since there are but N vertices, this process must halt before K = N+1. When it halts, we will have found a vertex of degree 1 (or a cycle, which is not possible).

2. How many trailing zeroes are there in the expansion of 100! (A computer solution will earn no credit.) For example, 10! has two trailing zeroes. (*Extra credit:* Same question for 1000!)

Solution: Since the only way that a trailing 0 can appear is to have factors 2 and 5. Clearly the number of factors 2 is at least as great as the number of 5s. So we must compute the highest power of 5 that is a factor of 100!
This amounts to considering the multiples of 5 that are no bigger than 100, namely: 5, 10, 15, 20, 25, 30, ..., 100.
Now each member of this sequence, with the exception of 25, 50, 75 and 100, (which are divisible by 5^2) contributes one factor of 5. Also each of 25, 50, 75 and 100 contributes two factors of five.
Hence the number of trailing zeros of 100! is 20 + 4 = 24.

3. Prove by mathematical induction that $n! < n^n$ for $n \geq 2$.

Solution: For $n \geq 2$ let $\mathcal{H}_n$ *be the statement* $n! < n^n$.

Base case: *Consider* $\mathcal{H}_2$ *:* $2! < 2^2$. *LHS = 2! = 2; RHS =* $2^2 = 4$. *Since LHS < RHS, the base case,* $\mathcal{H}_2$ *, is true.*

Inductive step: *Let k be a* **given** *integer that is greater than or equal to 2. We assume that* $\mathcal{H}_k$ *is true, namely:* $k! < k^k$.
Since k + 1 > 0: $(k+1)! < (k+1)\, k^k < (k+1)\, (k+1)^k = (k+1)^{k+1}$ *and so* $\mathcal{H}_{k+1}$ *is true.*

4. Let X, Y, Z be non-empty sets. Let f: X → Y be injective and let g: Y → Z be injective. Must it follow that the composition of the two functions g∘f : X →Z be injective? Recall the definition of composition, *viz.*
∀ x ∈X g∘f (x) = g(f(x)).
Give proof or counterexample.

Solution: Yes, it must follow that g∘f : X →Z is injective.

Proof: Assume that there exist a, b ∈ X such that g∘f *(a) =* g∘f *(b).*
That is, g(f(a)) = g(f(b)). Now since g is injective f(a) = f(b).
Since g is injective, it follows that a = b.
Thus g∘f (x) *is injective.*

5. Is the converse to problem # 4 true? *Give proof or counterexample.*
(Begin by stating the converse!)

Solution:

The converse states:
Let X, Y, Z be non-empty sets. Let f: X → Y and g: Y → Z.
Assume that g∘f : X → Z is injective. Then it follows that both f and g are injective.

The converse is false; here is a ***counterexample****:*

Let $X = \{1, 2\}$, $Y = \{1, 2, 3\}$ *and* $Z = \{1, 2\}$.

Define $f: X \to Y$ *as follows:* $f(1) = 1$ *and* $f(2) = 2$.
Define $g: Y \to Z$ *as follows:* $g(1) = 1$, $g(2) = 2$ *and* $g(3) = 1$.

Notice that g *is not injective since* $g(1) = g(3)$.
However $g \circ f : X \to Z$ *is injective since:*
$g \circ f(1) = 1$
$g \circ f(2) = 2$

Induction makes you feel guilty for getting something out of nothing, and it is artificial, but it is one of the greatest ideas of civilization.

- Herbert Wilf