MATH 201 Solutions: Test I

**Part I**  *[*2 pts for *each answer]*

1. Let G = (V, E) be a graph. G is said to be *connected* if

*any two vertices of the graph can be joined by a path (walk).*

G is said to be *acyclic* if *G has no cycles.*

The *degree* of a vertex is *the number of edges that emanate from the vertex.*

1. Let *p* and *q* be integers and let *m* be a positive integer. Then we write

*p q (mod m)* *if p – q is a multiple of m.*

1. Let X and Y be non-empty sets. Let f: X Y be a well-defined function. Then

*f is said to be injective if a, bX f(a) = f(b) a = b.*

*Or equivalently, a, bX a b f(a) f(b)*

f is said to be *surjective* if  *yY xX such that f(x) = y.*

f is said to be *bijective* if *f is both injective and surjective.*

1. Let V and W be non-empty sets. Let G: V W be a well-defined function.

Assume that V has an internal binary operation, denoted by ☺, and W has an

internal binary operation, denoted by ☹. Then G is said to be an *isomorphism*

 if *G: V W is a bijection*

and  *a, b V f(a ☺ b ) = f(a) ☹ f(b).*

1. Let R be a relation on a set X.

R is *reflexive* if  *a X aRa*

R is *symmetric* if  *a, b X aRb bRa*

R is *transitive* if  *a, b, c X aRb and bRc aRc.*

R is an *equivalence relation* if *R is reflexive, symmetric and transitive.*

1. Let Z be a set. A *partition* of Z is *a set of subsets {Pj} of Z satisfying the two conditions:*

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**Part II** [4 pts each]

1. Unlucky Ursula believes that the “cancellation law”

*ca cb (mod n) implies that a b (mod n)*

is always valid as long as *c* is not congruent to 0 mod n.

Show that, for *n = 8*, Ursula is *not* correct. That is, find integers a, b, c that provide a counter-example for the case *mod 8*.

*Solution: Here is a counterexample:*

*Let a = 2, b = 4 and c = 4.*

*Then ca = 8 and cb = 16. So ca cb 0 (mod 8)*

*Furthermore, c is not congruent to 0 (mod 8)*

*Yet a is not congruent to b (mod 8)*

2. Happy Huskey believes he can construct a connected graph that has at least 3 vertices for which the degree of every vertex is odd. *Is such a graph possible?*

If so, exhibit one; if not, explain why not.

*Solution: Consider the complete graph on 4 vertices. Then the degree of each vertex is 3. Hence Happy will not be able to perform this task.*

1. Let L be the set of all current Loyola students. Define a relation ~ on L as follows:

For a, b L, a ~ b if *a* and *b* have the same blood type. Is this an equivalence relation? Briefly justify your answer.

*Solution:*

*Reflexive, since each person has the same blood type as him/her self. So a ~ a, for all students in the set L.*

*Symmetric:*

*Suppose students a and b in L have the same blood type. Then certainly b and a have the same blood type since nothing has changed but the order of the students.*

*So a ~ b b ~ a*

*Transitive:*

*Let a, b and c be students in L such that a ~ b and b ~ c.*

*Then both a and c must have the same blood type as b. Thus a ~ c.*

*So ~ is transitive since a, b, c X a ~ b and b ~ c a ~ c.*

*Hence, we have shown that ~ is an equivalence relation on L.*

4. Let X, Y be sets satisfying |X| = |Y| = 8. *How many* bijections f: X Y exist? Briefly justify your answer.

*Solution: This is an application of the product theorem. Assume that X is ordered.*

*Now the first member of X can be sent to any of the 8 members of Y.*

*Now, since f is injective (since it is given that b is a bijection), the second member of X can be sent to any one of 7 elements of Y.*

*Continue this process until all 8 members of X have been assigned values in Y.*

*The number of choices we have is 8!*

*5.* Let X, Y be sets satisfying |X| = 5 and |Y| = 8. *How many* surjections f: X Y exist? Briefly justify your answer.

*Solution: There are no surjections f: X Y.*

*Since the image of each element of X must be sent to* ***one and only one*** *member of Y (since f must be well-defined),the range of f must consist of 5 or fewer members of Y.*

1. Let X, Y be sets satisfying |X| = 7 and |Y| = 9. *How many* injections f: X Y exist? Briefly justify your answer.

*Solution: This is similar to problem 4. Order the set X. Then the first element of X can be sent to any of the 9 elements of Y. The second element can be sent to any one of the remaining 8 elements of Y. Continue this pattern until there are no new members of X.*

*Thus the number of injections f: X Y is 9 (8) (7) (6) (5) (4) (3) = (9!) / 2.*

7. Let P(X) be the power set of a non-empty set X. For any two subsets A and B of X, define the relation A *R* B to mean that A B = .
Justify your answer to each of the following?

*Solutions:*

**Is *R* reflexive?**

No: Let A be any non-empty subset of X (we know A exists because we were told that X is non-empty). Clearly A A = A .

**Symmetric?**

Yes: Let A and B be two subsets of X satisfying the condition A B = .

By commutativity of intersection, B A =A B = . Thus B is related to A.

**Transitive?**

No. Here is a counter-example.

Let X= {1, 2, 3}. Let A = {1, 2}, B = {3, 4}, and C = {2}.

Then A is related to B, B is related to C, yet A and C are not related since A C = {2}.

8. Let P(Z) be the power set of a non-empty set Z. For any two subsets A and B of Z, define the relation A ~ B to mean that A B.
Justify your answer to each of the following:

*Solution:*

*Is ~* ***reflexive****?*

*Yes, since, for any subset A of Z, A A.*

***Symmetric****?*

*No: Here is a counterexample:*

*Let Z = {1, 2, 3, 4}, let A = {1, 2} and let B = {1, 2, 3}. Then A is a subset of B, yet B is not a subset of A.*

***Transitive?***

*Yes: Suppose that A, B, C are subsets of Z satisfying:*

*A B and B C.*

*Then clearly A C*

9. Meek Max believes that he has constructed a connected graph of SEVEN vertices for which the degrees of the 5 vertices are given by 2, 3, 4, 5, 6 respectively. Could Max be correct or is he deluding himself? Justify your answer.

*Solution: Max cannot be correct, since no vertex in a graph of 5 vertices can have degree larger than 4.*

10. Assertive Albertine claims that the system of two equations

36x – 5y = 68

14x + 73 y = 2015

has no integer solutions. Using modular arithmetic (not algebra) show that Albertine is correct. (*Hint:* consider the equations mod 2.)

*Solution:*

*Consider the first equation mod 2: 0x + y 0, or simply y 0 mod 2*

*Now the second equation mod 2: 0x + y 1 or simply y 1 mod 2.*

*This is not possible since 0 certainly is not congruent to 1 (mod 2).*

11. Boastful Boris claims that 310 1 (mod 11). Is he correct? Justify your answer *without* using a calculator!

*Solution: Boris is correct:*

*32 = 9*

*34 = ( 32 )2 = 92 4 mod 11*

*38 = (34)2 42 5 mod 11*

*Now 310 = 38 32 (5)(9) = 45 1 mod 11*

 *“I was x years old in the year x2.”*

* Augustus de Morgan, when asked about his age