MATH 201 SOLUTIONS: TEST I

Part I *[2 pts for each answer]*

1. Let G = (V, E) be a graph. G is said to be *connected* if

   any two vertices of the graph can be joined by a path (walk).

   G is said to be *acyclic* if *G has no cycles.*

   The *degree* of a vertex is *the number of edges that emanate from the vertex.*

2. Let p and q be integers and let m be a positive integer. Then we write

   $p \equiv q \pmod{m}$ *if* $p - q$ *is a multiple of* m.

3. Let X and Y be non-empty sets. Let f: X → Y be a well-defined function. Then

   f is said to be injective if $\forall a, b \in X \quad f(a) = f(b) \Longrightarrow a = b$.

   Or equivalently, $\forall a, b \in X \quad a \neq b \Longrightarrow f(a) \neq f(b)$

   f is said to be *surjective* if $\forall y \in Y\ \exists x \in X$ *such that* $f(x) = y$.

   f is said to be *bijective* if *f is both injective and surjective.*

4. Let V and W be non-empty sets. Let G: V → W be a well-defined function. Assume that V has an internal binary operation, denoted by ☺, and W has an internal binary operation, denoted by ☹. Then G is said to be an *isomorphism*

   if *G: V → W is a bijection*

   and $\forall a, b \in V$ *f(a ☺ b) = f(a) ☹ f(b).*

5. Let R be a relation on a set X.

R is *reflexive* if $\forall a \in X \;\; aRa$

R is *symmetric* if $\forall a, b \in X \;\; aRb \Longrightarrow bRa$

R is *transitive* if $\forall a, b, c \in X \;\; aRb$ *and* $bRc \Longrightarrow aRc$.

R is an *equivalence relation* if *R is reflexive, symmetric and transitive.*

6. Let Z be a set. A *partition* of Z is *a set of subsets* $\{P_j\}$ *of Z satisfying the two conditions:*

$$\bigcup_j P_j = Z \quad and \quad \forall i \neq j, \; P_i \bigcap P_j = \phi.$$

Part II [4 pts each]

1. Unlucky Ursula believes that the "cancellation law"

$$ca \equiv cb \pmod n \text{ implies that } a \equiv b \pmod n$$

is always valid as long as *c* is not congruent to 0 mod n.
Show that, for $n = 8$, Ursula is *not* correct. That is, find integers a, b, c that provide a counter-example for the case *mod 8*.

Solution: Here is a counterexample:

Let $a = 2$, $b = 4$ *and* $c = 4$.
Then $ca = 8$ *and* $cb = 16$. *So* $ca \equiv cb \equiv 0 \pmod 8$
Furthermore, c is not congruent to 0 (mod 8)
Yet a is not congruent to b (mod 8)

2. Happy Huskey believes he can construct a connected graph that has at least 3 vertices for which the degree of every vertex is odd. *Is such a graph possible?*
If so, exhibit one; if not, explain why not.

Solution: Consider the complete graph on 4 vertices. Then the degree of each vertex is 3. Hence Happy will not be able to perform this task.

3. Let L be the set of all current Loyola students. Define a relation ~ on L as follows: For a, b ∈ L, a ~ b if *a* and *b* have the same blood type. Is this an equivalence relation? Briefly justify your answer.

Solution:

Reflexive, since each person has the same blood type as him/her self. So a ~ a, for all students in the set L.

Symmetric:
Suppose students a and b in L have the same blood type. Then certainly b and a have the same blood type since nothing has changed but the order of the students.
So $a \sim b \implies b \sim a$

Transitive:
Let a, b and c be students in L such that a ~ b and b ~ c.
Then both a and c must have the same blood type as b. Thus a ~ c.
So ~ is transitive since $\forall a, b, c \in X \quad a \sim b \text{ and } b \sim c \implies a \sim c$.

Hence, we have shown that ~ is an equivalence relation on L.

4. Let X, Y be sets satisfying |X| = |Y| = 8. *How many* bijections f: X → Y exist? Briefly justify your answer.

Solution: This is an application of the product theorem. Assume that X is ordered.
Now the first member of X can be sent to any of the 8 members of Y.
Now, since f is injective (since it is given that b is a bijection), the second member of X can be sent to any one of 7 elements of Y.
Continue this process until all 8 members of X have been assigned values in Y.
The number of choices we have is 8!

5. Let X, Y be sets satisfying |X| = 5 and |Y| = 8. *How many* surjections f: X → Y exist? Briefly justify your answer.

Solution: There are no surjections f: X → Y.

Since the image of each element of X must be sent to ***one and only one*** *member of Y (since f must be well-defined),the range of f must consist of 5 or fewer members of Y.*

6. Let X, Y be sets satisfying |X| = 7 and |Y| = 9. *How many* injections f: X → Y exist? Briefly justify your answer.

Solution: This is similar to problem 4. Order the set X. Then the first element of X can be sent to any of the 9 elements of Y. The second element can be sent to any one of the remaining 8 elements of Y. Continue this pattern until there are no new members of X.
Thus the number of injections f: X → Y is 9 (8) (7) (6) (5) (4) (3) = (9!) / 2.

7. Let P(X) be the power set of a non-empty set X. For any two subsets A and B of X, define the relation A *R* B to mean that A ∩ B = ∅.
Justify your answer to each of the following?

Solutions:

Is *R* reflexive?

No: Let A be any non-empty subset of X (we know A exists because we were told that X is non-empty). Clearly A ∩ A = A ≠ ∅.

Symmetric?

Yes: Let A and B be two subsets of X satisfying the condition A ∩ B = ∅.
By commutativity of intersection, B ∩ A = A ∩ B = ∅. Thus B is related to A.

Transitive?

No. Here is a counter-example.
Let X= {1, 2, 3}. Let A = {1, 2}, B = {3, 4}, and C = {2}.
Then A is related to B, B is related to C, yet A and C are not related since A ∩ C = {2}.

8. Let P(Z) be the power set of a non-empty set Z. For any two subsets A and B of Z, define the relation A ~ B to mean that $A \subseteq B$.
Justify your answer to each of the following:

Solution:

*Is ~ **reflexive**?*

Yes, since, for any subset A of Z, $A \subseteq A$.

***Symmetric**?*

No: Here is a counterexample:

Let Z = {1, 2, 3, 4}, let A = {1, 2} and let B = {1, 2, 3}. Then A is a subset of B, yet B is not a subset of A.

Transitive?

Yes: Suppose that A, B, C are subsets of Z satisfying:

$A \subseteq B$ and $B \subseteq C$.
Then clearly $A \subseteq C$

9. Meek Max believes that he has constructed a connected graph of SEVEN vertices for which the degrees of the 5 vertices are given by 2, 3, 4, 5, 6 respectively. Could Max be correct or is he deluding himself? Justify your answer.

Solution: Max cannot be correct, since no vertex in a graph of 5 vertices can have degree larger than 4.

10. Assertive Albertine claims that the system of two equations

$$36x - 5y = 68$$

$$14x + 73y = 2015$$

has no integer solutions. Using modular arithmetic (not algebra) show that Albertine is correct. (*Hint:* consider the equations mod 2.)

Solution:

Consider the first equation mod 2: $0x + y \equiv 0$, *or simply* $y \equiv 0 \mod 2$

Now the second equation mod 2: $0x + y \equiv 1$ *or simply* $y \equiv 1 \mod 2$.

This is not possible since 0 certainly is not congruent to 1 (mod 2).

11. Boastful Boris claims that $3^{10} \equiv 1 \pmod{11}$. Is he correct? Justify your answer *without* using a calculator!

Solution: Boris is correct:

$3^2 = 9$
$3^4 = (3^2)^2 = 9^2 \equiv 4 \mod 11$
$3^8 = (3^4)^2 \equiv 4^2 \equiv 5 \mod 11$

Now $3^{10} = 3^8\ 3^2 \equiv (5)(9) = 45 \equiv 1 \mod 11$

"I was x years old in the year x^2*."*

– Augustus de Morgan, when asked about his age