



*Gregor Samsa awoke one morning from uneasy dreams he found himself transformed in his bed into a gigantic insect.*

- Franz Kafka, **The Metamorphosis**

1. The Lucas sequence,  $L(n)$ , is defined recursively as follows:

$$L(1) = 2; \quad L(2) = 1; \quad L(n+1) = L(n) + L(n-1) \quad \text{for } n \geq 2.$$

[5 pts] List the first 10 Lucas numbers, i.e.  $L(1), L(2), \dots, L(10)$

*Answer: 2, 1, 3, 4, 7, 11, 18, 29, 47, 76*

**Extra Credit:** Prove, using “strong induction” that  $L(n+2) = F(n+2) + F(n)$  for  $n \geq 1$ .

Recall the principle of “strong induction”:

Let  $H(n)$  be a sequence of statements for  $n \geq 1$

Base case: show  $H(1)$  is true

Induction step: To Prove: For given  $n \geq 1$ , if  $H(1), H(2), \dots, H(n)$  are true, then  $H(n+1)$

*Solution:*

*Base case: Let  $n = 1$ . Then  $L(1+2) = L(3) = 3$ ;  $F(1+2) + F(1) = F(3) + F(1) = 2 + 1 = 3$*

*Let  $n = 2$ : Then  $L(2+2) = L(4) = 4$ ;  $F(2+2) + F(2) = F(4) + F(2) = 3 + 1 = 4$*

*So  $H_n$  is true for  $n = 1$  and  $n = 2$ .*

**Inductive step:** *Let  $n \geq 2$  be given and assume that  $H_k$  is true for  $k = 1, 2, \dots, n$*

Since we have assumed that  $H_n$  and  $H_{n-1}$  are valid (using the fact that  $n \geq 2$ ), we can use the inductive hypotheses:

$$L(n+2) = F(n+2) + F(n); \quad L(n+1) = F(n+1) + F(n-1)$$

Now, by definition of the Lucas sequence:  $L(n+3) = L(n+2) + L(n+1) =$

$$F(n+2) + F(n) + F(n+1) + F(n-1) =$$

$$F(n+2) + F(n+1) + F(n) + F(n-1) =$$

$$F(n+3) + F(n+1)$$

$$\text{Hence } L(n+3) = F(n+3) + F(n+1)$$

2. [5 pts] Consider a standard deck of 52 cards. In how many ways can one be dealt a 7-card hand (without regard to order) that contains no Queens and no Spades?

*Solution: There are 36 cards which are neither spades nor queens.*

*So our answer is  $\binom{36}{7}$*

3. [5 pts] How many non-negative integer solutions are there to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 99?$$

*Solution: Using the method of stars and bars, we have 4 bars and 99 stars. So the number of solutions is*

$$\binom{103}{4}$$

(b) [5 pts] Same question as (a), but now assume that the solution must consist of *positive* integers.

*Solution: First remove five ones, so that each  $x_i$  can be incremented by 1 once a solution is found. Again using the method of stars and bars, we have 4 bars and 94 stars. So the number of solutions is*

$$\binom{98}{4}$$

**Extra credit:** Same question as (a) except at least one of the components of a solution  $(x_1, x_2, x_3, x_4, x_5)$

must be 0. For example,  $97 + 1 + 0 + 0 + 1 = 99$  is one such solution.

**Solution:** Let  $A_i$  be the set of all solutions for which  $x_i = 0$ .

Then we need to compute  $|\bigcup_{n=1}^5 A_i|$ . Use PIE.

4. (a) [5 pts] Give the combinatorial definition of  $\binom{n}{r}$  — — —  
*not* the means of calculating this number or any factorials.

This represents the number of ways one can select a subset of size  $r$  from a set of size  $n$ .

(Using the word “subset” signifies that order does not matter.)

- (b) [5 pts] Explain briefly, *using only a combinatorial interpretation*, why

$$\binom{1776}{99} = \binom{1776}{1677} \quad (\text{Use complete sentences.})$$

The subsets of size 99 of the 1776 objects is in one-to-one correspondence with the subsets of size 1677 from the set of 1776. (Each subset  $A$  of size 99 can be mapped to  $X \sim A$ , that has 1677 members. This is a bijection.)

- (c) [5 pts] Explain briefly, *using only a combinatorial interpretation*, why

$$\binom{1789}{25} = \binom{1788}{25} + \binom{1788}{24} \quad (\text{Us complete sentences.})$$

Let  $*$  be a particular member of the set,  $X$ , of 1789 objects. The set of subsets of size 25 of  $X$  can be split into two disjoint sets,  $C$  and  $D$ .

$C$  is the set of all subsets of size 25 of  $X$  that include  $*$ .

$D$  is the set of all subsets of size 25 of  $X$  that exclude  $*$ .

Now  $C$  is in one-to-one correspondence with the subsets of  $X$  of size 25 that do include  $*$ .

And  $D$  is in one-to-one correspondence with all subsets of  $X$  of size 25 that do not include  $*$ .

So  $C$  equals  $\binom{1788}{24}$  and  $D$  equals  $\binom{1788}{25}$

Since C and D are disjoint, we achieve the desired result.

5. (a) [5 pts] State the *binomial theorem*.

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

(b) [5 pts] Using the binomial theorem, what is the coefficient (in terms of combinations) of  $x^{1789}y^{226}$  in the expansion of  $(x + y)^{2015}$ . (Express your answer in terms of combinations.)

Answer:  $\binom{2015}{1789}$

(c) [5 pts] Compute the value of  $\sum_{k=0}^n (-1)^k \binom{n}{k}$ . (Hint: Consider applying the binomial theorem.)

$$0 = (1 - 1)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k}$$

6. [5 pts] In the land of Oz it has been decreed that every password must be exactly 11 characters in length, contain *exactly one* capital letter (A, B, C..., Z), *exactly one* lower case letter (a, b, c, ..., z), and *exactly one* of three special characters (#, \$, @). The remaining characters must be digits (0, 1, 2, ..., 9). How many such passwords exist?

Solution: Choose the three places of 11 that will receive the Capital letter, Lower case letter, and Special character. These 3 places can be arranged in 3! Ways.

Next, each of the remaining 8 spots can be assigned any one of 10 digits.

So the total number of passwords is  $\binom{11}{3} 3! (10^8) 26^2(3)$

*Extra credit:* Albertine, an Ozian, wants her password to contain digits only in strictly increasing order. So, for example, 01A23a11\$87 would not be acceptable, but 02s34J5789@ is acceptable.

How many passwords are acceptable to Albertine?

7. [5 pts] How many ways are there to seat 14 people, consisting of 7 couples, in a row of seats (14 seats wide) if all couples demand adjacent seats?

Solution: "Glue" each of the couples together, so now we can arrange the 7 glued couples in 7! Ways. Now each couple can be glued in any one of 2 ways.

Thus the total number of ways to seat the 14 people is  $(7!) 2^7$ .

8. [5 pts] How many numbers between 1 and 200 (inclusive) are divisible by 2, 3 or 5. (Do *not* solve this by “brute force.”)

Solution: Use PIE to obtain 146.

9. [5 pts] We are given 46 balls: 10 black, 11 white, 12 red and 13 blue. In how many ways can you choose 9 balls such that exactly 3 are red?

Solution: We can use stars and bars here: After taking 3 reds (no choice here), there are 6 balls left, to be chosen from 3 boxes: Black, White, Blue

Using 2 bars and 6 stars, we obtain  $\binom{8}{2}$  choices.

10. [5 pts] Find a closed form expression for the sequence:

1, 3, 7, 13, 21, 31, 43, ...

Solution: Method 1: Observe that  $n(n - 1) + 1$  works.

Method 2: To realize that the formula is quadratic, look at successive differences:

2, 4, 6, 8, 10, 12, ...

Once again: 2, 2, 2, 2, 2, ...

So we can generate our sequence with a function of the form  $f(n) = an^2 + bn + c$

Plugging in (1, 2), (2, 4), (3, 6) we solve for a, b, c.

11. (Actuarial exam problem)

[5 pts] The Legrandin insurance company has 10,000 policyholders. Each policyholder is classified as (i) young or old; (ii) male or female; and (iii) married or single. Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company’s policyholders are young, female, and single?

(A) 280

(B) 423

(C) 486

(D) 880

(E) 896

**Solution:** Using Venn diagrams, or the following analysis, we find that (D) 880 is the correct answer.

$$\begin{aligned} |\text{Young} \cap \text{Female} \cap \text{Single}| &= |\text{Young} \cap \text{Female}| - |\text{Young} \cap \text{Female} \cap \text{Married}| = \\ &= |\text{Young}| - |\text{Young} \cap \text{Male}| - \{|\text{Young} \cap \text{Married}| - |\text{Young} \cap \text{Male} \cap \text{Married}|\} = \\ &= 3000 - 1320 - (1400 - 600) = 880. \end{aligned}$$