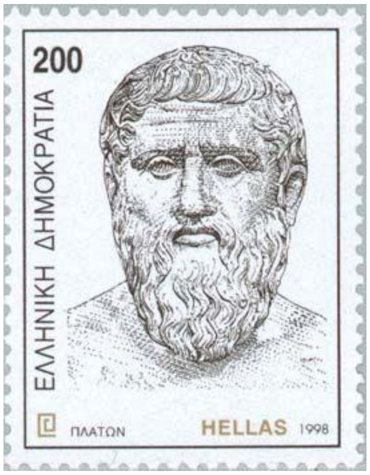
MATH 201 SOLUTIONS: TEST 3-A (IN CLASS)

Numbers are the highest degree of knowledge. It is knowledge itself.
- Plato



Part I *[7 pts each]*

1. Carefully state the *Well-Ordering Principle*.

*The **well-ordering principle** states that every non-empty set of positive integers contains a least element.*

2. Carefully state the *Euclidean Division Algorithm*.

Given two integers a and b, with $b \neq 0$, there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < |b|$.

3. Define *gcd(a, b).*

*The **greatest common divisor** of two integers (not both zero) is the largest integer which divides both of them.*

Equivalently, if a and b are not both zero, d = gcd(a, b) if the following two conditions are satisfied:

(1) $d|s$ and $d|b$

(2) If $e|a$ and $e|b$ then $|e| \leq d$

4. State (the conclusion of) Euclid's *extended gcd algorithm.*

*The conclusion of the **extended Euclidean algorithm** is:*

If a and b are integers, not both 0, then there exist integers x and y such that $ax + by = gcd(a, b)$.

5. Carefully state *Fermat's little theorem*.

If p is a prime number, then for any integer a,
$a^p \equiv a \pmod{p}$.

*If a is not divisible by p, **Fermat's little theorem** is equivalent to the statement that*
$a^{p-1} \equiv 1 \pmod{p}$.

6. State *Euclid's theorem* on prime numbers.

There exist infinitely many primes.

Part II *[10 pts each]*

1. Explain why every integer can be expressed in the form 5n, 5n+1, 5n+2, 5n+3 or 5n+4.

   It follows from Euclid's division algorithm that every integer can be represented as 5q+ r, where $0 \le r < 4$.

2. Using the Euclidean algorithm, find gcd(306, 657)

   gcd(306, 657) = gcd(657, 306) = gcd(45, 306) = gcd(306, 45) = gcd(36, 45) = gcd(45, 36) = gcd(9, 36) = gcd(36, 9) = gcd(0, 9) = gcd(9, 0) = ***9***

3. Using the extended Euclidian algorithm, find integers *x* and *y* such that 56x + 22y = gcd(56, 22).

   First we use the Euclidean algorithm to find gcd(56, 22):

   $56 = 22\,(2) + 12$

   $22 = 12(1) + 10$

   $12 = 10(1) + 2$

   $10 = 2(5) + 0$

   So the gcd is 2.

   Now, using back-substitution:

   $2 = 12 – 10(1)$

   $= 12 – (22 – 12) = 2(12) – 22$

   $= 2(56 – 22(2)) – 22 = 2(56) – 5\,(22)$

   We conclude that an integer solution of 56x + 22y = gcd(56, 22) is ***x = 2*** *and* ***y = -5***.

4. Prove that gcd(a, b – a) = gcd(a, b).

   Let d = gcd(a, b – a)   and d = gcd(a, b).*

Now $d|a$ and $d|\ (b-a)$ by definition of gcd.

So $d|\ \{a + (b-a)\} = b$

Thus $d|a$ and $d|\ b$. So, by definition of gcd, $|d| \leq |d^*|$

Next, $d^*|a$ and $d^*|b$. So, $d^*|\ ((-1)\ a + b) \implies d^* \mid b-a$.

Thus $|d^*| \leq |d|$.

So we arrive at $|d^*| = |d|$. Of course, d and d^* are each positive, so $d^* = d$.

5. Using Fermat's little theorem find 5^{101} (mod 31)

Since 31 is a prime and not a factor of 5, Fermat's little theorem states $5^{30} \equiv 1 \pmod{31}$.

And so $5^{90} = (5^{30})^3 \equiv 1^3 = 1 \pmod{31}$.

Next $5^{101} = 5^{90}\, 5^{11} \equiv 5^{11} \pmod{31}$.

Note that $5^3 = 125 = 4(31) + 1 \equiv 1 \pmod{31}$.

Finally, $5^{101} \equiv 5^{11} = (5^3)^3\, 5^2 \equiv 1^3\, 25 = \mathbf{25} \pmod{31}$.

6. The converse to Fermat's little theorem is false. Namely:

If $a^{m-1} \equiv 1$ mod m, it need not follow that *m* is prime.

(a) *[7 pts]* Find 2^{560} mod 561

First, note that $2^{10} \equiv 463 \pmod{561}$.

So $2^{20} = (2^{10})^2 \equiv 463^2 \equiv 67 \pmod{561}$

So $2^{40} = (2^{20})^2 \equiv 67^2 \equiv 1 \pmod{561}$

Finally, $2^{560} = (2^{40})^{14} \equiv 1^{14} = 1 \pmod{561}$

(b) *[3 pts]* Show that 561 is not a prime number. (Such numbers are called *pseudo-primes*.)

Since $3|561$, 561 cannot be prime.

7. Prove that if a|b and c|d then ac|bd.

Since $a|\ b$ $\exists m \in Z$ such that $b = am$.

Since $c|d\ \exists n \in Z$ *such that* $d = cn$.

Thus $bd = (am)(cn) = (ac)(mn)$. *Of course* $mn \in Z$.

Hence $ac|bd$.

8. Prove that $\sqrt{3}$ is irrational.

Suppose, contrary to fact, that that $\sqrt{3}$ *is rational. Then* $\exists a, b \in Z, b \neq 0$ *, such that*

$\sqrt{3} = a/b$.

We may assume that a and b are relatively prime. (If not, divide each of a and b by gcd(a, b).)

So $a^2 = 3b^2$. *Hence* a^2 *is a multiple of 3. This implies that a is a multiple of 3. (Examine the three cases:* $a = 3p$, $a = 3p+1$, $a = 3p+2$.*)*

Hence $\exists q \in Z$ *such that* $a = 3q$.

So $3b^2 = a^2 = (3q)^2 = 9q^2$.

From this, we obtain: $b^2 = 3q^2$. *As argued earlier, this implies that b is a multiple of 3.*

This is clearly a contradiction, since if a and b were divisible by 3, then a and b would not be relatively prime, as we assumed above.

9. Prove that the *square of any integer* is either of the form 3k or 3k+1.

Using the division algorithm, every integer, n, may be expressed as

$$n = 3z + r \text{ where } r = 0, 1, 2.$$

Examining each of these three cases:

$$(3z)^2 = 3(3z^2)$$

$$(3z + 1)^2 = 9z^2 + 6z + 1 = 3(3z^2 + 2z) + 1$$

$$(3z + 2)^2 = 9z^2 + 12z + 4 = 3(3z^2 + 4z + 1) + 1$$

Thu,s for each of the three cases, n^2 *is either of the form 3k or 3k+1.*

EXTRA CREDIT:

1. *[10 pts]* Prove by induction: For $n \in N$, if $a^n|b^n$ then $a|b$.

For each $n \in N$, let H_n represent the statement: if $a^n|b^n$ then $a|b$.

Base Case: *H_1 is true since if $a^1|b^1$ then clearly $a|b$.*

Inductive step: *Let $n \geq 0$ be given. Assume that $a^{n+1}|b^{n+1}$.*

Let $d = gcd(a, b)$. Let $A = a/d$ and $B = b/d$. We have proven earlier that A and B are relatively prime. Now, $a^{n+1}|b^{n+1}$ implies that $A^{n+1}|B^{n+1}$.

It is easy to show that A^{n+1} and B^{n+1} are relatively prime.

Rewriting: $AA^n \mid BB^n$

Then, by Euclid's lemma, since A and B are relatively prime, $A^n|B$ or $A^n|B^n$.

If $A^n|B^n$, then we can use the inductive hypothesis to conclude that $A|B$ and hence $a|b$.

If $A^n \mid B$, then of course $A|B$.

2. *[10 pts]* Prove that $(3n)!/(3!)^n$ is an integer for all $n \geq 0$. (Recall that $0! = 1$)

For each $n \geq 0$, let H_n represent the statement: if $a^n|b^n$ then $a|b$.

Base case: *$n = 0$: $(3(0))!/(3!)^0 == 1 \in Z$.*

Inductive step: *Assume that $n \geq 0$ is given and that H_n is true.*

Now $(3(n+1))!/(3!)^{n+1} = (3n+3)!/(3!)^{n+1}$ $= \left(\frac{(3n)!}{(3!)^n}\right)\left(\frac{(3n+1)(3n+2)(3n+3)}{3!}\right) =$

$$\left(\frac{(3n)!}{(3!)^n}\right)(n+1)\frac{(3n+1)(3n+2)}{2}$$

Now, by inductive hypothesis, $\left(\frac{(3n)!}{(3!)^n}\right)$ *is an integer. Furthermore, the product of two consecutive integers is even. Thus $(3n+1)(3n+2)$ is divisible by 2.*

So we have shown that H_{n+1} is true.