# Class discussion: Cardinality

revised

**14 November 2017**



**I** What does it mean to say that two sets have the *same cardinality*? What does it mean to say that a set is *countably infinite*?

 **II** Show that each of the following sets is countable:

1. The set of non-negative integers.

(b)     The set of integers greater than or equal to 13.

(c)    **Z**

(d)    The set of positive even integers.

(e)     The set of even integers.

(f)     The set of odd integers.

(g) The set of rational numbers strictly between 0 and 1.

**III** (a) Show that a subset of a countable set is either finite or countable.

(b) Show that if *A* and *B* are disjoint countable sets then so is the union of *A* and *B*. What if *A* and *B* are not disjoint?

(c) Show that if *A* and *B* are countable sets then so is the Cartesian product of *A* and *B*.

(d) Prove that a countable union of countable sets is countably infinite.

(e) Prove that the set of rational numbers strictly between 0 and 1 is uncountable.

(f) Demonstrate that **Q** is countable.

**IV** Show that if S is a collection of sets, then cardinality is an equivalence relation on S.

 **V** Using Cantor’s diagonal argument, prove that **R** is not countable.

 

**VI** (a) Let *X* be a set. Recall the definition of the power set, P *(X)*, of *X*.

1. Show that the power set of a finite set is finite. In such case, describe the relationship between |X| and | P (X)|.
2. Let X = {a, b, c, d} and let F: X → P (X) be defined by:

F(a) = {a, c, d}, F(b) = {a, d}, F(c) = φ, F(d) = {d}

Find D\* = $\left\{j\in X| j\notin F(j)\right\}$

1. Let X = Z+ and let G: X → P(X) be defined by:

G(a) = {all prime numbers, p, such that a ≤ p ≤ 2a}

Find D\* = $\left\{j\in X| j\notin G(j)\right\}$

1. Let X = **Q** and let H: X → P (X) be defined by:

$H\left(z\right)= $ {all prime numbers, *p*, such that z ≤ p ≤ 2z}

Find D\* = $\left\{q\in X| q\notin H(q)\right\}$

1. Let X = **R** and let V: X → P (X) be defined by:

$V(a)=\left\{\begin{array}{c}\left\{0\right\} if a\leq 0 \\\left(a, a+1\right) if a\in Q^{+}\~Z \\\left[a-1, a\right] if a is a positive irrational number\\\left\{a, a+3\right\} if a\in Z^{+} \end{array}\right.$

Is V well-defined? If so, find D\* = $\left\{z\in X| z\notin V(z)\right\}$

1. Prove *Cantor’s Theorem*: X and P(X) are not of the same cardinality.

**Highly recommended:**

MIT lecture notes on cardinality, 24.118 (paradox and infinity)



[Georg Ferdinand Ludwig Cantor](http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Cantor.html) (1845 – 1918) is best known for

his discovery of transfinite numbers and the creation of Set Theory.

*Lenore nodded. ‘Gramma really likes antinomies. I think this guy here, ’looking down at the drawing on the back of the label, ‘is the barber who shaves all and only those who do not shave themselves’.*

- David Foster Wallace, **The Broom of the System**

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