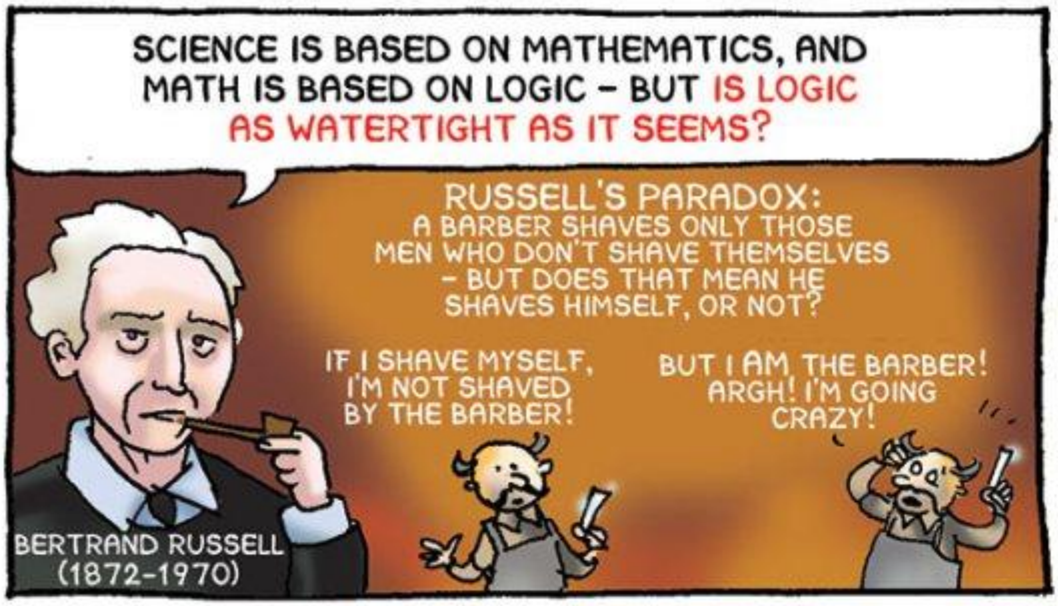
CLASS DISCUSSION: CARDINALITY

REVISED

14 November 2017



I What does it mean to say that two sets have the *same cardinality*? What does it mean to say that a set is *countably infinite*?

II Show that each of the following sets is countable:

(a) The set of non-negative integers.

(b) The set of integers greater than or equal to 13.

(c) **Z**

(d) The set of positive even integers.

(e) The set of even integers.

(f) The set of odd integers.

(g) The set of rational numbers strictly between 0 and 1.

III (a) Show that a subset of a countable set is either finite or countable.

(b) Show that if *A* and *B* are disjoint countable sets then so is the union of *A* and *B*. What if *A* and *B* are not disjoint?

(c) Show that if *A* and *B* are countable sets then so is the Cartesian product of *A* and *B*.

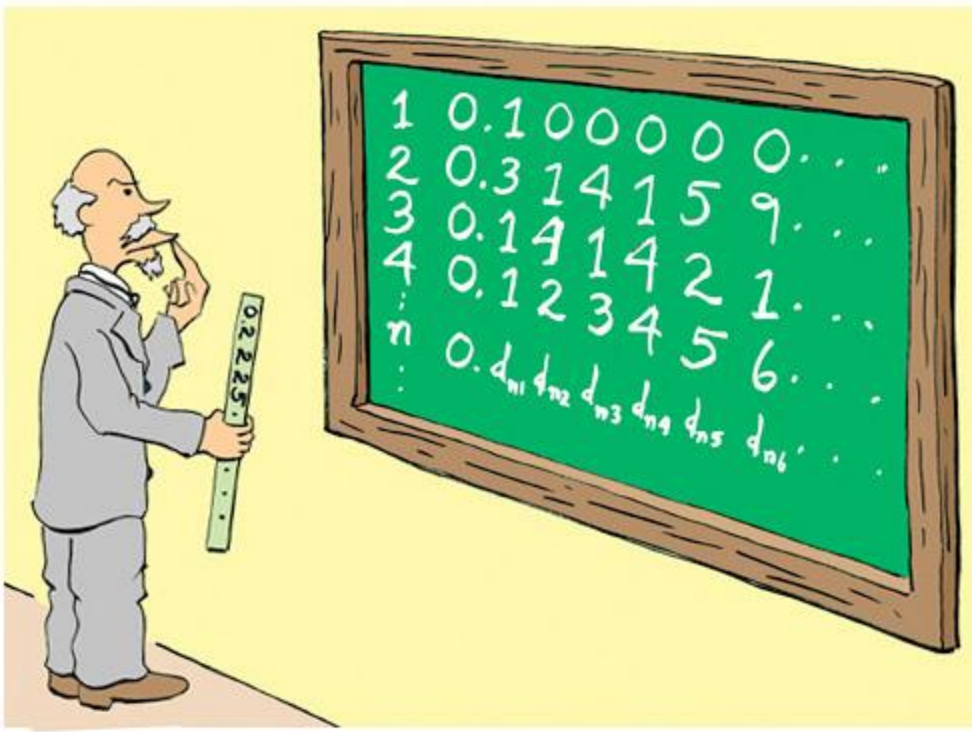
(d) Prove that a countable union of countable sets is countably infinite.

(e) Prove that the set of rational numbers strictly between 0 and 1 is uncountable.

(f) Demonstrate that **Q** is countable.

IV Show that if S is a collection of sets, then cardinality is an equivalence relation on S.

V Using Cantor's diagonal argument, prove that **R** is not countable.



VI (a) Let *X* be a set. Recall the definition of the power set, $\mathscr{P}(X)$, of *X*.

(b) Show that the power set of a finite set is finite. In such case, describe the relationship between |X| and $|\mathscr{P}(X)|$.

(c) Let X = {a, b, c, d} and let F: X → $\mathscr{P}$(X) be defined by:

F(a) = {a, c, d}, F(b) = {a, d}, F(c) = φ, F(d) = {d}

Find $D^* = \{j \in X \mid j \notin F(j)\}$

(d) Let X = Z^+ and let G: X → P(X) be defined by:

G(a) = {all prime numbers, p, such that a ≤ p ≤ 2a}

Find $D^* = \{j \in X \mid j \notin G(j)\}$

(e) Let X = **Q** and let H: X → $\mathscr{P}$(X) be defined by:

$H(z) =$ {all prime numbers, *p*, such that z ≤ p ≤ 2z}

Find D* = $\{q \in X |\ q \notin H(q)\}$

(f) Let X = **R** and let V: X → $\mathscr{P}$(X) be defined by:

$$V(a) = \begin{cases} \{0\}\ if\ a \leq 0 \\ (a, a+1)\ if\ a \in Q^{+} {\sim} Z \\ [a-1, a]\ if\ a\ is\ a\ positive\ irrational\ number \\ \{a, a+3\}\ if\ a \in Z^{+} \end{cases}$$

Is V well-defined? If so, find D* = $\{z \in X |\ z \notin V(z)\}$

(g) Prove *Cantor's Theorem*: X and $\mathscr{P}$(X) are not of the same cardinality.

Highly recommended:

MIT lecture notes on cardinality, 24.118 (paradox and infinity)

Georg Ferdinand Ludwig Cantor (1845 – 1918) is best known for his discovery of transfinite numbers and the creation of Set Theory.

Lenore nodded. 'Gramma really likes antinomies. I think this guy here, 'looking down at the drawing on the back of the label, 'is the barber who shaves all and only those who do not shave themselves'.

- David Foster Wallace, **The Broom of the System**