

## CLASS DISCUSSION: CLOSURE PROPERTIES

21 NOVEMBER 2017

**I** For each of the following, decide if the given set is *closed* under the operations of addition, subtraction, multiplication, and/or division. Give counterexample or proof.

1.  $\mathbf{Z}$ , the set of integers
2.  $\mathbf{N}$ , the set of natural numbers
3.  $\mathbf{Q}$ , the set of rational numbers
4.  $\mathbf{R}$ , the set of real numbers
5.  $\mathbf{C}$ , the set of complex numbers
6.  $\mathbf{R} \sim \{0\}$
7.  $\mathbf{Q} \sim \{0\}$
8. the set of even integers
9. the set of even positive integers
10. the set of odd integers
11. the set of odd positive integers
12. the set of prime numbers
13. the set of composite integers  $\geq 2$
14. the set of irrational numbers
15. the set of integers that are perfect squares
16. the set of integers of the form  $3k$
17. the set of integers of the form  $5k+1$
18. the set of Fibonacci numbers
19. given two integers  $a$  and  $b$ , not both 0, let  $S = \{ax + by \mid x, y \in \mathbf{Z}\}$

**II** (a) Let  $S = \mathbf{N} \times \mathbf{N}$ . Define the following operation,  $\odot$ , on  $S$ :  $(a, b) \odot (c, d) = (a+c, b+d)$   
Is  $S$  closed under  $\odot$ ? Is  $\odot$  associative? commutative?

(b) Let  $S = \mathbf{Q} \times \mathbf{R}$ . Define the following operation,  $\oplus$ , on  $S$ :

$$(a, b) \oplus (c, d) = (ac, a + b + c + d)$$

Is  $S$  closed under  $\oplus$ ? Is  $\oplus$  commutative?

(c) Let  $S = \mathbf{Q}$ . Define the following operation,  $\spadesuit$ , on  $S$ :

$$a \spadesuit b = 13 + ab$$

Is  $S$  closed under  $\spadesuit$ ? Is  $\spadesuit$  associative? commutative?

**III** Determine which of the following sets are *closed* under the given operation. Explain. Unless otherwise stated, the operation is addition of functions.

1. The set of all continuous functions  $f: \mathbf{R} \rightarrow \mathbf{R}$
2. The set of all differentiable functions  $f: \mathbf{R} \rightarrow \mathbf{R}$
3. The set of all polynomials of degree 8.
4. The set of all polynomials of degree  $\leq 5$
5. The set of all polynomials,  $p(x)$ , such that  $p(0) = 0$
6. The set of all non-negative continuous functions that are defined on the interval  $[0, 1]$ .
7. The set of all points in the first quadrant: that is,  $V = \{(a, b) \mid a \geq 0, b \geq 0\}$  with the usual addition.
8. The set of all points in the first and third quadrants: that is,  
 $V = \{(a, b) \mid a \geq 0, b \geq 0\} \cup \{(a, b) \mid a \leq 0, b \leq 0\}$  with the usual addition.
9. The set of all  $3 \times 3$  diagonal matrices with the usual matrix addition.
10. The set of all differentiable functions,  $f(x)$ , defined on the real line such that  $f'(4) = 1$
11. The set of all differentiable functions,  $f(x)$ , defined on the real line such that  $f'(9) = 0$
12. The set of all convergent sequences with the usual addition.
13. The set of all sequences having finitely many non-zero terms
14. The set of all sequences having finitely many zero-terms
15. Let  $S$  be the set of all real sequences that have both an infinite number of negative terms and an infinite number of positive terms. For example, the sequence  
 $(1, -1, 2, -2, 3, -3, 4, -4, \dots) \in S$  but  $(1, 2, 3, 4, 5, \dots) \notin S$
16. The set of all  $2 \times 2$  matrices that have zero determinant, under the operation of matrix addition.
17. The set of all  $2 \times 2$  matrices that have zero determinant, under the operation of matrix addition.
18. The set of all  $2 \times 2$  matrices that have non-zero determinant, under the operation of matrix addition.
19. The set of all  $10 \times 10$  matrices that have at least 50 entries of 0, under the operation of matrix addition.
20. The set of all  $10 \times 10$  matrices that have only positive entries, under the operation of matrix addition.

*To some extent the beauty of number theory seems to be related to the contradiction between the simplicity of the integers and the complicated structure of the primes, their building blocks. This has always attracted people.*

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