CLASS DISCUSSION: 12 OCTOBER 2017

Part I:

1. State the well-ordering principle.
2. Using the WOP, prove the division algorithm.

Part II:

3. Let a, b be integers. Then a$a \equiv b \ (mod\ 6)\ if\ and\ only\ if\ a \equiv b\ (mod\ 2)\ and\ a \equiv b\ (mod\ 3)$
4. Prove by contradiction that $\sqrt{3}$ *is irrational.*
5. Prove by contradiction that if x is irrational then so is $x^{1/2}$.
6. Prove by contradiction that $2^{1/3}$ is irrational.
7. Suppose n ∈ Z. Prove that n is odd if and only if n^2 is odd.
8. Prove by contradiction that if $0 \leq t \leq \pi/2$ then cos t + sin t ≥ 1.
9. Let a and b be integers. If $a^2 + b^2 = c^2$, then a or b is even.

7. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

8. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

9. Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

10. There exist no integers a and b for which $21a + 30b = 1$.

11. There exist no integers a and b for which $18a + 6b = 1$.

12. For every positive $x \in \mathbb{Q}$, there is a positive $y \in \mathbb{Q}$ for which $y < x$.

13. For every $x \in [\pi/2, \pi]$, $\sin x - \cos x \geq 1$.

14. If A and B are sets, then $A \cap (B - A) = \emptyset$.

15. If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

16. If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

17. For every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.

18. Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.

Prove the following statements using any method from Chapters 4, 5 or 6.

19. The product of any five consecutive integers is divisible by 120. (For example, the product of 3,4,5,6 and 7 is 2520, and $2520 = 120 \cdot 21$.)

20. We say that a point $P = (x, y)$ in $\mathbb{R}^2$ is **rational** if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathbb{Q}^2$. An equation $F(x, y) = 0$ is said to have a **rational point** if there exists $x_0, y_0 \in \mathbb{Q}$ such that $F(x_0, y_0) = 0$. For example, the curve $x^2 + y^2 - 1 = 0$ has rational point $(x_0, y_0) = (1, 0)$. Show that the curve $x^2 + y^2 - 3 = 0$ has no rational points.

21. Exercise 20 (above) involved showing that there are no rational points on the curve $x^2 + y^2 - 3 = 0$. Use this fact to show that $\sqrt{3}$ is irrational.

22. Explain why $x^2 + y^2 - 3 = 0$ not having any rational solutions (Exercise 20) implies $x^2 + y^2 - 3^k = 0$ has no rational solutions for k an odd, positive integer.

23. Proof or counter-example: if a $ac \equiv bc\ (mod\ n)\ \ then\ a \equiv b\ (mod\ n)$

Exercises for Chapter 7

Prove the following statements. These exercises are cumulative, covering all techniques addressed in Chapters 4–7.

1. Suppose $x \in \mathbb{Z}$. Then x is even if and only if $3x+5$ is odd.
2. Suppose $x \in \mathbb{Z}$. Then x is odd if and only if $3x+6$ is odd.
3. Given an integer a, then a^3+a^2+a is even if and only if a is even.
4. Given an integer a, then a^2+4a+5 is odd if and only if a is even.
5. An integer a is odd if and only if a^3 is odd.
6. Suppose $x, y \in \mathbb{R}$. Then $x^3+x^2y = y^2+xy$ if and only if $y = x^2$ or $y = -x$.
7. Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2+y^2$ if and only if $x = 0$ or $y = 0$.
8. Suppose $a, b \in \mathbb{Z}$. Prove that $a \equiv b \pmod{10}$ if and only if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.
9. Suppose $a \in \mathbb{Z}$. Prove that $14 \mid a$ if and only if $7 \mid a$ and $2 \mid a$.
10. If $a \in \mathbb{Z}$, then $a^3 \equiv a \pmod{3}$.
11. Suppose $a, b \in \mathbb{Z}$. Prove that $(a-3)b^2$ is even if and only if a is odd or b is even.
12. There exist a positive real number x for which $x^2 < \sqrt{x}$.
13. Suppose $a, b \in \mathbb{Z}$. If $a+b$ is odd, then a^2+b^2 is odd.
14. Suppose $a \in \mathbb{Z}$. Then $a^2 \mid a$ if and only if $a \in \{-1, 0, 1\}$.
15. Suppose $a, b \in \mathbb{Z}$. Prove that $a+b$ is even if and only if a and b have the same parity.
16. Suppose $a, b \in \mathbb{Z}$. If ab is odd, then a^2+b^2 is even.
17. There is a prime number between 90 and 100.
18. There is a set X for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.
19. If $n \in \mathbb{N}$, then $2^0+2^1+2^2+2^3+2^4+\cdots+2^n = 2^{n+1}-1$.
20. There exists an $n \in \mathbb{N}$ for which $11 \mid (2^n - 1)$.
21. Every real solution of $x^3+x+3 = 0$ is irrational.
22. If $n \in \mathbb{Z}$, then $4 \mid n^2$ or $4 \mid (n^2-1)$.
23. Suppose a, b and c are integers. If $a \mid b$ and $a \mid (b^2 - c)$, then $a \mid c$.
24. If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$.