# Class Discussion: Proof by Contradiction; 5 Oct.

## Method

In order to prove a proposition *P* by contradiction:

1. Write, “We use proof by contradiction.”
2. Write, “Suppose *P* is false.”
3. Deduce a logical contradiction.
4. Write, “This is a contradiction. Therefore, *P* must be true.”

## Example

Remember that a number is *rational* if it is equal to a ratio of integers. For example, 3*.*5 = 7*/*2 and 0*.*1111…= 1*/*9 are rational numbers. On the other hand, we’ll prove by contradiction that$ \sqrt{2}$ is irrational.

**Proposition:** $\sqrt{2}$ *is* *irrational.*

*Proof.* We use proof by contradiction. Suppose the claim is false; that is, $\sqrt{2}$ is rational.

Then we can write 2 as a fraction *a/b* in *lowest* *terms*.

Squaring both sides gives 2 = *a*2*/b*2 and so 2*b*2 = *a*2.

This implies that *a* is even; that is, *a* is a multiple of 2.

Therefore, *a*2 must be a multiple of 4. Because of the equality 2*b*2 = *a*2, we know 2*b*2 must also be a multiple of 4.

This implies that *b*2 is even and so *b* must be even. But since *a* and *b* are both even, the fraction *a/b* is not in lowest terms.

This is a contradiction. Therefore, $\sqrt{2}$ must be irrational.

## Potential Pitfall

Often students use an indirect proof when a direct proof would be simpler. Such proofs aren’t wrong; they just aren’t excellent. Let’s look at an example. A function *f* is *strictly* *increasing* if *f*(*x*) *> f*(*y*) for all real *x* and *y* such that *x > y*.

**Theorem.** *If* *f and* *g are* *strictly* *increasing* *functions,* *then* *f* + *g is* *a* *strictly* *increasing* *function.*

Let’s first look at a simple, direct proof.

*Proof.* Let *x* and *y* be arbitrary real numbers such that *x > y*. Then:

|  |  |
| --- | --- |
| *f*(*x*) *> f*(*y*)  | (since *f* is *strictly* increasing)  |
| *g*(*x*) *> g*(*y*)  | (since *g* is *strictly* increasing)  |

Adding these inequalities gives:

*f*(*x*) + *g*(*x*) *> f*(*y*) + *g*(*y*)

Thus, *f* + *g* is strictly increasing as well.

Now we *could* prove the same theorem by contradiction, but this makes the argument needlessly convoluted.

*Proof.* We use proof by contradiction. Suppose that *f* + *g* is not strictly increasing. Then there must exist real numbers *x* and *y* such that *x > y*, but

*f*(*x*) + *g*(*x*) ≤ *f*(*y*) + *g*(*y*)

This inequality can only hold if either *f*(*x*) ≤ *f*(*y*) or *g*(*x*) ≤ *g*(*y*). Either way, we have a contradiction because both *f* and *g* were defined to be strictly increasing. Therefore, *f* +*g* must actually be strictly increasing.

**Exercise1:** If ais even then a2 is even. Prove by contradiction.

If a, b ∈ Z and a ≥ 2, then a - b or a - (b +1).

**Exercise2:** If a, b ∈ Z and a ≥ 2, then $a∤b or a∤(b+1)$

Study the following proof (from our textbook). Is it logically correct?



**Exercises:**

1. Prove by contradiction that there exists no largest even integer.
2. Prove by contradiction that $\sqrt[3]{1332}>11.$
3. There exist no integers a and b for which 21a+30b = 1.
4. Prove by contradiction that there exists no smallest positive real number.
5. Prove by contradiction that there exists no largest prime number. (Euclid’s proof)
6. Prove by contradiction that $\sqrt{3} is irrational.$
7. Prove by contradiction that if x is irrational then so is x1/2.
8. Prove by contradiction that 21/3 is irrational.
9. Suppose n ∈ Z. If n is odd, then n2 is odd.
10. Suppose n ∈ Z. If n2 is odd, then n is odd.
11. Prove by contradiction that if 0 ≤ t ≤ /2 then cos t + sin t ≥ 1.
12. Let *n* be a positive integer. Prove that log2 n is rational if and only if n is a power of 2.
13. Prove the arithmetic – geometric mean inequality by contradiction.
14. Employing the method of proof by contradiction show that for any non-degenerate triangle (that is, every side has positive length), the length of the hypotenuse is *less than* the sum of the lengths of the two remaining sides.
15. Let a and b be integers. If a2 + b2 = c2, then a or b is even.
16. Prove that there are infinitely many prime numbers (Euclid).



[G. H. Hardy](https://en.wikipedia.org/wiki/G._H._Hardy) described proof by contradiction as "one of a mathematician's finest weapons", saying "It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game."

- G. H. Hardy, **A Mathematician’s Apology**

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