MATH 201 SOLUTIONS: TEST I 21 SEPTEMBER 2017

Instructions: Answer any 10 of the 12 problems. You may answer more for extra credit.

1. Let p = "Aldo is Italian" and q = "Frederick is English"
Write each of the following as a statement in sentential logic.

(a) Aldo is not Italian.

$\sim p$

(b) Aldo is Italian while Frederick is English.

$p \wedge q$

(c) If Aldo is Italian then Frederick is not English.

$p \Rightarrow \sim q$

(d) Aldo is Italian or if Aldo isn't Italian then Frederick is English.

$p \vee (\sim p \Rightarrow q)$

(e) Either Aldo is Italian and Frederick is English, or neither Aldo is Italian nor Frederick is English.

$(p \wedge q) \vee \sim(p \vee q)$

Equivalently:

$(p \wedge q) \vee (\sim p \wedge \sim q)$

2. Four hungry children are waiting for a snack.

(a) In how many ways can 13 *different* candy bars be distributed amongst the 4 children?

Solution: Each candy bar can be given to any one of the 4 children. This is repeated a total of 13 times. Using the multiplication principle, we find the number of ways this can be accomplished is
4^{13}.

(b) In how many ways can 13 Kit Kats (*indistinguishable*) be distributed amongst the 4 children?

Solution: Here we use 13 stars representing the 13 indistinguishable candy bars. Since there are 4 children, we need 3 bars to act as separators. Now the number of "code words" using 13 stars and 3 bars is

$$\binom{16}{3}$$

3. Find the flaw in the following bogus proof. Explain!

- Step 1: Let $a = b$.
- Step 2: Then $a^2 = ab$
- Step 3: Then $a^2 + a^2 = a^2 + ab$
- Step 4: Then $2a^2 = a^2 + ab$
- Step 5: Then $2a^2 – 2ab = a^2 + ab – 2ab$
- Step 6: Then $2a^2 – 2ab = a^2 – ab$
- Step 7: This can be written as $2(a^2 – ab) = 1(a^2 – ab)$
- Step 8: Dividing each side by $(a^2 – ab)$ yields $1 = 2$

Solution: Here we divided by 0 in Step 8, since a = b (from Step 1) implies that $(a^2 – ab) = 0$.

4. Introducing appropriate predicates, express each of the following statements in first-order predicate logic. Assume that the universe of discourse is the set of all people who live in Illinois.

Solution: Let W(x) mean “x walks” and T(x) means “x talks.”

(a) Someone walks and talks.

$\exists x\ \ W(x) \land T(x)$

(b) Someone walks and someone talks.

$\exists x\ \exists y\ W(x) \land T(y)$

(c) Everyone who walks is unable to talk.

$\forall x\ W(x) \Rightarrow \sim T(x)$

(d) No one who talks can walk.

$\forall x\ \ T(x) \Rightarrow \sim W(x)$ or equivalently $\forall x\ \ \sim T(x) \land \sim W(x)$

(e) No one can both walk and talk.

$\forall x \ \sim(T(x) \wedge W(x))$ or equivalently (using de Morgan's law) $\forall x \ (\sim T(x) \vee \sim W(x))$

5. Albertine has been dealt a hand of 8 cards from a standard deck of 52.

(a) In how many ways can the hand fail to contain a pair? (This means no two cards in the hand form a pair.)

Solution: First choose 8 ranks from the13 ranks; then choose a card from each of the chosen ranks.

Using the multiplication principle, the number of ways the hand can fail to contain a pair is

$\binom{13}{8}\binom{5}{1}$

(b) In how many ways can the hand consist of exactly 4 pairs (not allowing 4 of a kind)?

Solution: Choose 4 ranks. Then from each rank choose 2 cards. Thus:

$\binom{13}{4}\binom{4}{2}^4$

(c) In how many ways can the hand consist a flush? (Flush means that all cards are of the same suit.)

Solution: First choose a suit; then choose 8 cards from that suit:

$\binom{4}{1}\binom{13}{8}$

(d) In how many ways can the hand consist of three of a kind, three of a kind, and one pair?

Solution: First choose two ranks, those that will correspond to the three of kind. Then from each rank choose 3 cards. Finally choose a 3rd rank and select 2 cards from it.

$\binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{2}$

6. Use a truth table to determine if the following statement is always true:

$$\sim p \Rightarrow \big((p \wedge \sim q) \Rightarrow (p \wedge q)\big)$$

Solution:

p	q	~p	~q	$p \wedge \sim q$	$p \wedge q$	$(p \wedge \sim q) \Rightarrow (p \wedge q)$	$\sim p \Rightarrow ((p \wedge \sim q) \Rightarrow (p \wedge q))$
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	F	F	T	T
F	F	T	T	F	F	T	T

7. (a) Prove that A – B = $B^c - A^c$ using the technique of showing set equality developed in class.

Solution: **Part I**: The LHS is contained in the RHS. Let x$\in A - B$. Then by definition of relative complement, x$\in A \text{ and } x \notin B$. *Equivalently,* x$\notin A \text{ and } x \in B^c$. *Again, by defniition,* $x \in B^c - A^c$.

Part II: The RHS is contained in the LHS. Let x$\in B^c - A^c$. Then by definition of relative complement, x$\in B^c \text{ and } x \notin A^c$. *Equivalently,* x$\notin B \text{ and } x \in A$. *Again, by defniition,* $x \in A - B$.

(b) Using de Morgan's law, prove that $(A \cup (B \cap C))^c = A^c \cap (B^c \cup C^c)$.

(Note that this method is very short; any other method would be more time consuming.)

Solution: Invoking de Morgan's laws twice: $(A \cup (B \cap C))^c = (A^c \cap (B \cap C)^c) = A^c \cap (B^c \cup C^c)$.

8. In calculus we learn that a real-valued function *f* defined on the real line is defined to be ***continuous*** if: For every real number *p*, and for every $\varepsilon > 0$, there is a $\delta > 0$ such that |f(x) – f(p)| is less than ε whenever |x – p| is less than δ.

(a) Write this definition as a statement in first–order predicate logic. Be sure to specify the universe of discourse.

Solution: Let the universe of discourse consist of the set of all continuous functions that are defined on the real line. Then f is consists of the set of real numbers.
So f is continuous if:

$$\forall p\ \forall \epsilon > 0\ \exists \delta > 0\ \forall x\ \ |x - p| < \delta\ \Rightarrow |f(x) - f(y)| < \varepsilon$$

(b) Negate your definition in (a).

Solution:

$$\exists p\ \exists \epsilon > 0\ \ \forall\ \delta > 0\ \exists x\ \ (|x - p| < \delta\ ) \wedge (|f(x) - f(y)| \geq \varepsilon)$$

9. Let X = {cat, dog, llama, wombat}. Find each of the following cardinalities. (You need not justify your answers.)

(a) $|P(X)| = 2^4$

(b) $|P(X) - \varphi| = 2^4 - 1$

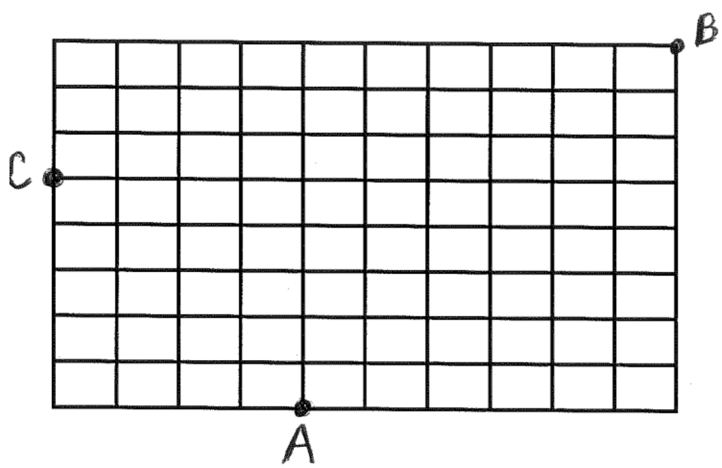
(c) $|P(X) \times \{cat, dog\}| = 2^4 (2) = 2^5$

(d) $|P(X) - \{cat, dog\}| = 2^4 - 1$

(e) $|P(X - \{llama\})| = 2^{4-1} = 2^3$

10. Vladimir, at point A, must meet Estragon at point B. But first Vladimir must stop at the hardware store, point C, to purchase some rope. In how many ways can Vladimir achieve this journey if he takes the *shortest* route in each of the two sub-trips?

Solution:



Solution: From A to C there are $\binom{9}{4}$ direct routes (since he must travel 4 blocks west and 5 blocks north).
From C to B, there are $\binom{13}{3}$ direct roots (since he must travel 10 blocks east and 3 blocks north).
Finally, using the Multiplication Principle, there are $\binom{9}{4}\binom{13}{3}$ possible routes from A to B via C.

11. (a) Is the following argument correct? Explain.

Lucky will buy a house only if Pozzo buys a car. Pozzo will buy a car only if Estragon buys a motorcycle. Estragon will not buy a motorcycle. So, Lucky will not buy a house.

Solution: The argument is correct:

Let L = "Lucky will buy house"; P = "Pozzo will buy a car"; and E = "Estragon will buy a motorcycle".

So we are given that

L only if P

P only if E

From this we obtain: L $\Rightarrow$ *P and P* $\Rightarrow$ *E,*

Thus L $\Rightarrow$ *E*. *So* $\sim E \Rightarrow \sim L$

(b) Write the *negation* of the following sentence. Simplify as much as possible.

$$\exists p \in X\ \exists q \in B\ \forall r \in C\quad p \le r \le q$$

Solution: $\forall p \in X\ \forall q \in B\ \exists r \in C\quad (r < p) \vee (r > q)$

12. For each of the five logical formulas, indicate whether or not it is true when the domain of discourse is the set of non-negative integers, the integers, the rationals, or the reals.
For example, in the set of all non-negative integers, there does not exist a number whose square is two, but in the set of all real numbers there does exist a number whose square is 3.

Each section should have five T or F responses. No penalty for guessing.

❖ Domain of discourse is the set of *all non-negative integers*.

$\exists x\ (x^2 = 2)$ False

$\forall x\ \ \exists y\ (x^2 = y)$ True

$\forall y\ \ \exists x\ (x^2 = y)$ False

$\forall x \neq 0\ \ \exists y\ (xy = 1)$ False

$\exists x\, \exists y\ (x + 2y = 2) \wedge (x + 4y = 5)$ False

❖ Domain of discourse is the set of *all integers*.

$\exists x\ (x^2 = 2)$ False

$\forall x\ \ \exists y\ (x^2 = y)$ True

$\forall y\ \ \exists x\ (x^2 = y)$ False

$\forall x \neq 0\ \ \exists y\ (xy = 1)$ False

$\exists x\, \exists y\ (x + 2y = 2) \wedge (x + 4y = 5)$ False

❖ Domain of discourse is the set of *all rationals*.

$\exists x\ (x^2 = 2)$ False

$\forall x\ \ \exists y\ (x^2 = y)$ True

$\forall y\ \ \exists x\ (x^2 = y)$ False

$\forall x \neq 0 \;\; \exists y \;\; (xy = 1)$ True

$\exists x \, \exists y \;\; (x + 2y = 2) \wedge (x + 4y = 5)$ True

❖ Domain of discourse is the set of *all reals.*

$\exists x \;\; (x^2 = 2)$ True

$\forall x \;\; \exists y \;\; (x^2 = y)$ True

$\forall y \;\; \exists x \;\; (x^2 = y)$ False

$\forall x \neq 0 \;\; \exists y \;\; (xy = 1)$ True

$\exists x \, \exists y \;\; (x + 2y = 2) \wedge (x + 4y = 5)$ True