**MATH 201 Solutions: TEST III 30 November 2017**

Instructions: Answer any 11 of the following 14 problems. You may answer more than 11 to earn extra credit.

1. Let A, B, and C be non-empty finite sets. Suppose that |A| = 44 and |B| = 55, and |C| = 2. Then answer each of the following question. (You need not show your work; the answer will suffice.)
2. The number of mappings from A into B is 5544.

*Here we use the multiplication principle.*

 *The number of surjections from A into B is 0 .*

*The range of any mapping from A into B cannot contain more than 44 elements.*

1. The number of injections from A into B is, depending upon your point of view, is P(55, 44) or

*Each element of A can “choose” one of the remaining elements of B. In particular, the first element of A selects from 55 elements; the second element of A can choose from 54 elements; etc.*

1. The number of surjections from B into C is 244 – 2.

*Here we subtract from the total number of mappings the two that have range consisting of only one element.*

1. The number of injections from B into A is 0 .

*Pigeon-hole principle*

1. The number of bijections from A into A is 44!
2. Let F: X →Y be a mapping and let G: Y →Z be a mapping. Assume that
3. Prove that G must be surjective.

*Proof: Let*

*Now, let q*

*Hence G: Y →Z is surjective.*

1. By exhibiting a counter-example, show that F *need not* be surjective.

(*Hint:* You may wish to assume that X, Y and Z are finite sets of small cardinalities. However, this is not essential to produce a counter-example.)

*Counter-example: Let X = {1}, Y = {2, 3}, Z = {4}.*

*Define G: Y →Z by: G(2) = G(3) = 4.*

*Define F: X →Y be defined by G(1) = 2.*

*Then it is easily seen that G is not surjective, yet is surjective.*

1. Let X = {0, 1, 2, 3, 4, 5} and Y = P(X). Define T: X →P(X) as follows:

T(0) = {1, 3}

T(1) = {4}

T(2) =

T(3) = {1, 3, 5}

T(4) = {0, 1, 2, 3, 4, 5}

T(5) = {4, 5}

Let D\* = {i

1. Find (explicitly) D\*.

*D\* = {0, 1, 2}*

1. Is T injective? Explain!

*Yes: The 6 outputs of T are distinct.*

1. Is T surjective? Explain!

*No: For example, {1, 2, 3} is not in the range of T.*

1. Is T bijective? Explain!

*No: Since T is not surjective.*

1. Find 5119 (mod 59). (Hint: Use Fermat’s little theorem.)

*Since 59 is a prime number, it follows from Fermat’s Little Theorem that 558 ≡ 1 (mod 59).*

*Thus 5119 = 5(2)(58) + 3 = (558)2 53 ≡ (1)2 53 = 125 = 2(59) + 7 ≡ 7 (mod 59).*

1. Let X be the set of all integers greater than 1.

If a, b define the following relation ⌛ on X: a⌛b if the largest prime factor of a is at least as big as the largest prime factor of b.

1. Is ⌛ reflexive? Give proof or counter-example.

*Yes: Clearly the largest prime factor of a is the largest prime factor of a. Hence a ⌛ a for all a X.*

1. Is ⌛ symmetric? Give proof or counter-example.

*No: Consider a = 5 and b = 7. Then a⌛ b is true but b ⌛ a is false.*

1. Is ⌛ transitive? Give proof or counter-example.

*Yes: If the largest prime factor of a is greater or equal to the largest prime factor of b, and the largest prime factor of b is greater or equal to the largest prime factor of c, then clearly the largest prime factor of a is greater than or equal to the largest prime factor of c. Thus ⌛ is transitive.*

1. Find 270 + 370 (mod 13). (*Hint:* Use Fermat’s little theorem.)

*Using Fermat’s Little Theorem, 212 ≡ 1 (mod 13) and 312 ≡ 1 (mod 13).*

*Now (mod 13) 270 + 370 = (212)5 210 + (312)5 310 ≡ 210 + 310 = (25)2  + (33)3  3 = 322 + 273 3 ≡ 62 + 3 = 39 ≡ 0 (mod 13)*

1. Let X = {1, 2, 3, 4, 5, 6}. Consider the following relation diagram on X.



1. Does this diagram define an equivalence relation on X? Explain.

*Yes: Let R denote the relation.*

*Then 1R1, 2R2, 2R2, 3R3, 4R4, 5R5, 6R6. Thus R is reflexive.*

*Also R is symmetric, since the diagram tells us that if aRb then bRa .*

*R is transitive by checking the above diagram. For example, if 1R2 and 2R3 then 1R3.*

Answer either (b) OR (c).

1. If the answer to (a) is negative, is the relation reflexive, symmetric, and/or transitive?
2. If the answer to (a) is affirmative, list the equivalence classes of X.

*We answer (c) : There are three equivalence classes:*

1. ***{1, 2, 3}***
2. ***{4, 5}***
3. ***{6}***
4. (a) Let X and Y be disjoint countably infinite sets. Prove that X∪Y is countable.

*Let X = {a1, a2, a3, a4, a5, a6, …} and Y = {b1, b2, b3, b4, b5, b6, …}.*

*Then we can enumerate X∪Y as follows:*

*X∪ Y = {a1, b1,a2, b2, a3, b3, a4, b4, a5, b5, a6, b6, …}*

*Thus X∪ Y is countably infinite.*

(b) Using part (a), prove that the set of irrational numbers is uncountable.

Let X = set of rational numbers and let Y = set of irrational numbers. Then *X∪ Y = R.*

*Since X is countable and R is uncountable, 8(a) implies that Y must be uncountable.*

1. Let Z be the set of all integers, and let X be the set of all polynomials in one variable with integer coefficients.

Define H: X→ Z as follows:

For p, let

1. Explain why H is well-defined?

*Yes, H is well-defined because for every polynomial p(x) with integer coefficients,*

*are integers, and the set of integers is closed under multiplication. So*

1. Is H surjective? Give proof or counter-example

*Yes: Let k Z. Define p(x) X as follows: For all x X let p(x) = k, the constant function.*

*Then H(p) = p(1) + 0 + 0 = k. So H is surjective.*

1. Is H injective? Give proof or counter-example.

No: Let p1(x) = 13, a constant function. Next, let p2(x) = 13 + (x – 1)4

Then p2 (1) = 13, . Thus and yet

So H is not injective.

1. For any real numbers, *c* and *d*, let us define the binary operation  as follows:

c  d = c2 d + c d2 + 1

Give either a *brief* justification or counterexample for each of the following:

1. The set of integers is closed under the operation  ?

*True: We know that Z is closed under multiplication, subtraction and addition. Hence, if c and d are integers, then c  d is an integer.*

1. The operation  is commutative: i.e., for all real numbers *c* and *d*,

c  d = d  c.

*True:*  c  d = *c2 d + c d2 + 1 = d2 c + d c2 + 1 =*d  c.

1. Let c, d∈**R**. If either *c* or *d* equals 0, then c  d = 1.

*True:*  *If either c = 0 or d = 0, then c2 d = c d2 = 0.*

*Thus* c  d = c2 d + c d2 + 1 = 0 + 0 + 1 = 1

1. Let c, d∈R. If c  d = 1 then either *c* or *d* equals 0.

*False: Let c = 1 and d = -1. Then c d =* c2 d + c d2 + 1 = (-1) + (1) + 1 = 1.

1. The set of irrational numbers is closed under .

*False: Let c = and d Then c d =* c2 d + c d2 + 1 = +

1. The set of odd integers is closed under .

*True: If c and d are odd integers, then c2 and d2 are also odd. Furthermore, c2 d and c d2 are odd since the set of odd numbers is closed under multiplication.*

*Thus c2 d + c d2 is even. Finally, c d = c2 d + c d2 + 1 = even + 1 = odd.*

11. Let X be the set of all continuous functions on the interval [0, 1].

For f, g X, define the relation

 f g if

1. Is Explain.

*Yes: , it follows that f f.*

1. Is Explain.

No: Let f = 1 be a constant function and let g = 0 be a constant function.

Then . Thus f

1. Is Explain.

*Yes: f. Then*

*Consequently,*

1. Let X be set of all 2 matrices whose entries are either 0 or 1.

Define M N if the sum of the four entries of M = the sum of the four entries of N. Show that this defines an equivalence relation on X. Find the equivalence classes of X.

*Solution: There are 16 2 binary matrices. Clearly M M for M*

*Now M N means that matrix M has the same number of 0s as N. Thus N M.*

*Next, if A B and B C then clearly A C.*

*There are 5 equivalence classes, viz.*

*Sum is 0:*

*Sum is 1:*

*Sum is 2:*

*Sum is 3:*

*Sum is 4:*

13. Let P(X) be the power set of a non-empty set X. For any two subsets A and B of X, define the relation A ⌛ B on P(X) to mean that A ∩ B = (the empty set).
Justify your answer to each of the following?

Is ⌛ reflexive? Explain.

*No: A ∩ A = implies that A = . This need not be true since X is non-empty.*

Is ⌛ symmetric? Explain.

*Yes: If A ⌛ B then A ∩ B = . From this it follows that B ∩ A = .*

*Consequently, B ⌛ A.*

Is ⌛ transitive? Explain.

*No: Let X. Such a p must exist since X is non-empty. Then P(X) must include and {p}. Let A = C = {p} and B = . Then A ⌛ B since A ∩ B = and B ⌛ C since*

*B ∩ C = . However, A ∩ C = {p}, so A ⌛ C is false.*

14. Let P(X) be the power set of a non-empty set X. For any two subsets A and B of X, define the relation A 🐈 B on P(X) to mean that A B. (That is: A is a subset of B, possibly A = B.) Justify your answer to each of the following:

Is 🐈 reflexive?

*Yes: For all A , A 🐈A since A A*

*.*

Is 🐈symmetric?

*No: Let A be the empty set and let B = X. Then A B but it is not the case that*

*B A. Hence A 🐈B but it is not the case that B A.*

Is 🐈transitive?

*Yes: If A B and B C, then A C.*

*Thus if A B and B C, then A C.*

If you’re never sorry

Then you can’t be forgiven

If you’re not forgiven

Then you can’t be forgotten

If you’re not forgotten

Then you can live forever

If you live forever

Then you’ll begin to dream

Of death…

 - Regina Spektor, **Pound of Flesh**

*Numbers are the highest degree of knowledge. It is knowledge itself.*

- Plato

