MATH 351: QUESTIONS FOR CLASS DISCUSSION, 26TH NOVEMBER

MAPPINGS; UNIFORM CONTINUITY

#GIVINGTUESDAY™

November 27, 2018

❖ **Continuous Functions on Compact Intervals** (sections 3.4, 13.5)

1. *(Review)* State the **Maximum Theorem** for continuous functions on compact sets.
2. *(Review)* Explain why the Maximum Theorem implies the Boundedness Theorem.
3. *(Review)* Give an example of a continuous function on (0, 1] with neither maximum nor minimum on this interval, but which does not have the limit ∞ $or -\infty$ $as\ x \to 0^+$.
4. *(Review)* What is meant by ***the image of S under the map f*** ?
5. Find f(S) for each of the following:

   (a) $f(x) = x^2$; S = (1, 4]

   (b) f(x) = ln x; S = [1, e].

   (c) f(x) = sin x; S = (π/4, 9π/4)

   (d) $f(x) = \sin\frac{1}{x}$; $S = (0, 1)$
6. State and prove the **Continuous Mapping Theorem**.
7. Find functions f(x) and g(x) such that

   $(a)\ f([-1, 1]) = (-1, 1)$

   $(b)\ g\left(\left[0, \frac{\pi}{2}\right]\right) = [0, \infty)$

   What general observation can you make about such functions *f* and *g*?
8. Does the Continuous Mapping Theorem imply both the IVT and the Maximum theorem?
9. (Abbott, Understanding Analysis) Give an example of each of the following, or provide a brief argument why the request is impossible.

   (a) A continuous function defined on [0, 1] with range (0, 1).

   (b) A continuous function defined on (0, 1) with range [0, 1].

   (c) A continuous function defined on (0, 1] with range (0, 1).

   (d) If f is continuous on a bounded set A, then f(A) is bounded.

   (e) If f is defined on R, and f(K) is compact whenever K is compact, then f is continuous on R.

(f) If f is continuous on [a, b], with f(x) > 0 for all $a \leq x \leq b, then \frac{1}{f}$ *is bounded on* $[a, b]$. (Meaning $\frac{1}{f}$ has bounded range.)

10. Define **uniform continuity**.
11. Prove, using only the definition, that f(x) = $x^2 + 1$ is uniformly continuous on [0, 2] but not uniformly continuous on [0, ∞].
12. Prove that g(x) = $\sqrt{x}$ is uniformly continuous on [0, ∞).
13. Show that g(x) = x^2 is *not* uniformly continuous on [0, ∞).
14. State the negation of uniform continuity (aka sequential criterion for absence of uniform convergence).
15. Show that $f(x) = \sin\frac{1}{x}$ *is* ***not*** *uniformly continuous* $(0, 1)$.
16. State and prove the **Uniform Continuity Theorem**.
17. Prove, using only the definition, that f(x) = $x^2 + 1$ is uniformly continuous on [0, 2] but *not* uniformly continuous on [0, ∞].