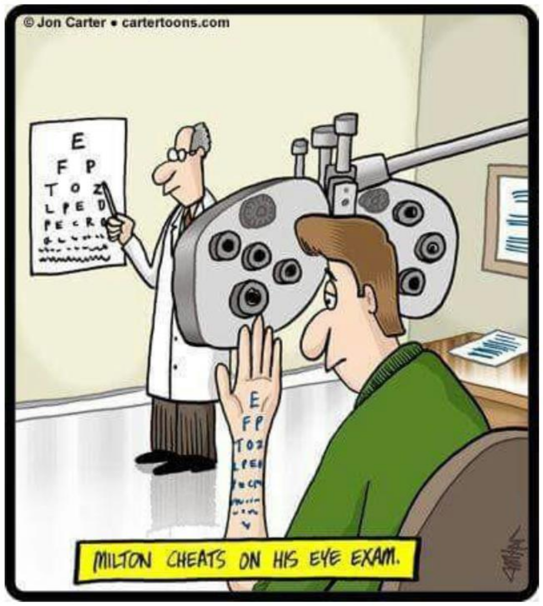
MATH 351: QUESTIONS FOR CLASS DISCUSSION, 5TH NOVEMBER

LOCATION THEOREMS; SEQUENTIAL CONTINUITY



1. (Review) [S. Abbott, **Understanding Analysis**, 2nd edition, Springer (2016)]
True or False? Justify!

(a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.

(b) If $\lim_{x\to b} f(x) = L$ and b happens to be in the domain of f, then L = f(b).

(c) If $\lim_{x\to b} f(x) = L, then \lim_{x\to b} 3\,(f(x)-2)^2 = 3(L-2)^2$

(d) If $\lim_{x\to b} f(x) = 0, then \lim_{x\to b} f(x)g(x) = 0,$ for any function g $(with\ domain\ equal\ to\ the\ domain\ of\ f).$

2. State the **error form for limits**: Let f(x) = L + e(x).

3. What is the relationship between continuity and limit? Define: limit as x $\to \infty$ *or* x $\to -\infty$.

4. Prove, using only the definition of limit, that $\lim_{x\to\infty} \frac{x^4 - x + \ln x + 99\sin 2018x}{(x-2)x^4 + 5x^3} = 0.$

5. Define infinite limits.

6. (Mattuck) Does "absolute continuity" imply continuity? That is, if |f(x)| is continuous on I, will f(x) be continuous on I?

7. [S. Abbott, **Understanding Analysis**, 2nd edition, Springer (2016)]

Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 but such that f(x)g(x) and f(x) + g(x) are continuous at 0.

(b) A function f(x) continuous at 0 and g(x) not continuous at 0 such that f(x) + g(x) is continuous at 0.

(c) A function f(x) continuous at 0 and g(x) not continuous at 0 such that f(x)g(x) is continuous at 0.

(d) A function f(x) not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.

(e) A function f(x) not continuous at 0 such that |f(x)| is continuous at 0.

8. State the **Algebraic Limit Theorems** for functions. What are the consequences for continuous functions?

9. Prove, using your result from (8) that every polynomial is continuous. Is the same true of rational functions?

10. State and prove the **Squeeze Theorem** for limits. What version applies to infinite limits?

11. State the **Limit Location Theorem** for functions.

12. State the **Function Location Theorem.**

13. *True or False:*

$$\lim_{x\to 5} f(x) = 0 \ \Rightarrow \lim_{x\to 5} \frac{1}{f(x)} = either\ \infty\ or -\infty.$$

14. Prove, using the Squeeze Theorem, that $x^{\frac{1}{n}} \to 1\ as\ x \to 1$

15. Let $f(x) = \int_1^x \frac{\sqrt{5+t}}{t} dt$. Prove that $f(x) \to \infty\ as\ x \to \infty$.

16. Marcel wishes to prove that if $\lim_{x\to\infty} f(x) = 0\ then \lim_{x\to\infty} f(x)\sin x = 0$.

He writes the following "proof":

$$-1 \le \sin x\ \le 1 \Rightarrow -f(x) \le f(x)\sin x \le f(x);\quad Now\ apply\ the\ Squeeze\ theorem.$$

Albertine, the grader, gives Marcel a grade of D for his proof. Why?

17. What is **Sequential Continuity**?

18. Sequential criterion for functional limits. The following are equivalent:

(a) $\lim_{x\to c} f(x) = L$

(b) For all sequences $\{a_n\}$ in the domain of f, satisfying $a_n \neq c\ and\ a_n \to c, it\ follows\ that\ f(a_n) \to L.$

19. State and prove the **Composition Theorem** for continuous functions.

20. Why is $\sin(x^3 + 2018)$ continuous everywhere?

21. Prove that if f is continuous then |f(x) is continuous as well.

22. Find four continuous functions y = f(x) satisfying $y^2 = x^2$.

23. Let $f(x) = \begin{cases} 1 & if\ x = \frac{1}{n} for\ n \in N \\ 0 & otherwise \end{cases}$. Prove, using the Sequential Continuity Theorem, that f(x) is discontinuous at x = 0.

24. [S. Abbott, **Understanding Analysis**, 2nd edition, Springer (2016)]

Let f be a function defined on R.

(a) Let's say f is onetinuous at c if for all $\varepsilon > 0\ we\ can\ choose\ \delta = 1\ and\ it\ follows\ that\ |f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuous on all of R.

(b) Let's say f is equaltinuous at c if for all $\varepsilon > 0\ we\ can\ choose\ \delta = \varepsilon\ and\ it\ follows\ that\ |f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on R but is nowhere onetinuous, or explain why there is no such function.