MATH 351: QUESTIONS FOR CLASS DISCUSSION, 9TH NOVEMBER

BOLZANO'S THEOREM

1. (Review) State and prove the **Squeeze Theorem** for limits. What version applies to infinite limits?

2. (Review) State the **Limit Location Theorem** for functions.

3. (Review) State the **Function Location Theorem.**

4. (Review) *True or False:*

$$\lim_{x\to 5} f(x) = 0 \ \Rightarrow \lim_{x\to 5}\frac{1}{f(x)} = either\ \infty\ or -\infty.$$

5. (Review) Prove, using the Squeeze Theorem, that $x^{\frac{1}{n}} \to 1\ as\ x \to 1$

6. (Review) Let $f(x) = \int_1^x \frac{\sqrt{5+t}}{t}dt$. Prove that $f(x) \to \infty\ as\ x \to \infty$.

7. (Review) Marcel wishes to prove that if $\lim_{x\to\infty} f(x) = 0\ then \lim_{x\to\infty} f(x)\sin x = 0$.

He writes the following "proof":

$$-1 \le \sin x\ \le 1 \Rightarrow -f(x) \le f(x)\sin x \le f(x);\quad Now\ apply\ the\ Squeeze\ theorem.$$

Albertine, the grader, gives Marcel a grade of D for his proof. Why?

8. (Review) What is **Sequential Continuity**? Is sequential continuity equivalent to continuity?

9. Sequential criterion for functional limits. The following are equivalent:

(a) $\lim_{x\to c} f(x) = L$

(b) For all sequences $\{a_n\}$ in the domain of f, satisfying $a_n \ne c\ and\ a_n \to c, it\ follows\ that\ f(a_n) \to L.$

10. State and prove the **Composition Theorem** for continuous functions.

11. Why is $\sin(x^3 + 2018)$ continuous everywhere?

12. Prove that if f is continuous then |f(x)| is continuous as well.

13. Find four continuous functions y = f(x) satisfying $y^2 = x^2$.

14. Write the negation of the statement: f(x) is continuous at x = p

15. Let $f(x) = \begin{cases} 1 & if\ x = \frac{1}{n}\ \ for\ \ n \in N \\ 0 & otherwise \end{cases}$

Prove, using the Sequential Continuity Theorem, that f(x) is discontinuous at x = 0.

16. [S. Abbott, **Understanding Analysis**, 2nd edition, Springer (2016)]

Let f be a function defined on **R**.

(a) Let's say f is onetinuous at c if for all $\varepsilon > 0\ we\ can\ choose\ \delta = 1\ and\ it\ follows\ that\ |f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuous on all of R.

(b) Let's say f is equaltinuous at c if for all $\varepsilon > 0\ we\ can\ choose\ \delta = \varepsilon\ and\ it\ follows\ that\ |f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on R but is nowhere onetinuous, or explain why there is no such function.

(c) Let's say f is lesstinuous at c if for all $\varepsilon > 0$ we can choose $0 < \delta < \varepsilon$ and it follows that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is lesstinuous on **R** that is nowhere equaltinuous, or explain why there is no such function.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

17. State and prove Bolzano's theorem for a continuous function on a compact interval.

18. Show that the Intermediate Value Theorem is a consequence of Bolzano's theorem.

19. What is the IVP?

20. Apply the intermediate value theorem to show that the equation $x^5 - 3x^2 = -1$ has a solution in the interval [0, 1].

21. Apply the intermediate value theorem to show that the equation $x^5 - 3x^2 + 3 = 0$ has a solution in the interval [−1, 1].

22. Apply intermediate value property to show that the equation $\sqrt{x^6 + 5x^4 + 9} = 3.5$ has a solution in the interval [0, 1].

23. Prove that f(x) = 9 sin x – x^5 = 1 has at least one solution.

24. State the Intersection Principle.

25. State the Intermediate Value Property.

26. Prove that if f(x) is strictly monotone and has the IVP on [a, b], then f is continuous on [a, b].