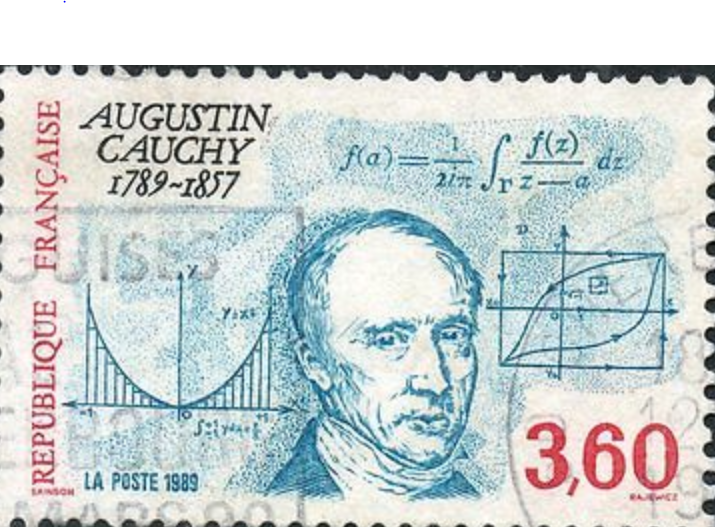
Math 351: class discussion, 10 October 2018



Cauchy sequences & supremum, infimum of sets

1. Review: What is the ***Cauchy criterion*** for convergence?
2. Given a sequence for all n. Must it follow that be Cauchy?
3. Show directly that

*Hint:* First note that

1. Consider the following statement: “Given for n >>1”

Give an *example* of an increasing sequence that satisfies this condition, yet is not a Cauchy sequence.

1. Given a sequence for all *n* and *k*. Prove that is Cauchy.
2. **Definitions:** Let
3. An ***upper bound*** for S is \_\_\_\_\_\_\_\_\_\_\_\_.
4. S is ***bounded above*** if \_\_\_\_\_\_\_\_\_\_\_\_.
5. The ***maximum*** of S is \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.
6. **Definition:** Let The ***supremum*** of S (abbreviated sup S, aka lub S) is \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.
7. Let Prove that:
8. If max S exists, then it is unique.
9. If sup S exists, then it is unique.
10. *[exercises from a graduate Finance program]*
11. State what it means for a sequence *not* to be Cauchy. Use quantifiers.
12. Prove that if is Cauchy then is also Cauchy.
13. Give an example of a Cauchy sequence such that is not Cauchy.
14. Prove the ***Completeness Property for sets***, *viz.*

If S is non-empty and bounded above, then sup S exists.

1. Introduce ***infimum*** of S (*aka* glb)
2. Is there any sequence of numbers a1, a2, . . . such that the set {a1, a2, . . .} is bounded, but the sequence has no maximal and no minimal elements?
3. Find the supremum for the following set and prove that your answer is correct. S =
4. Consider the set A = { (−1)n n : n ∈ N}. (a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
5. Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?
6. (Mattuck, Example 6.4) Consider the recursively defined sequence .

*Prove that this is a Cauchy sequence and determine its limit.*

1. Consider the set A = {x ∈ **R**: 1 < x < 2}.

(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?

(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

15. Prove that if S ⊂ R is non-empty and bounded below, then it has an infimum.

16. (UC, Berkeley) For S ⊂ R a non-empty subset that is bounded above and x ∈ R, let xS be the set {xs: s ∈ S}.

(a) Show that if x > 0, then sup (xS) = x sup (S).

(b) Show that if x < 0, then inf (xS) = x inf (S).

17. (UC, Berkeley) Let S, T **R** be non-empty subsets that are bounded from above, and define

S + T = {s + t: s ∈ S, t ∈ T}.

Show sup(S + T) = sup(S) + sup(T). Then, use this to prove that if x ∈ R and S +x is the set {s+x: s ∈ S}, then

sup(S + x) = sup(S) + x.

Additional Exercises (S. Abbott, **Understanding Analysis**, 2nd edition, Springer)

1. Decide whether each of the following statements is True or False. Provide either a brief justification or a counterexample.
2. If every proper subsequence of converges, then converges as well.
3. If
4. If

to different limits.

1. If .
2. If are Cauchy sequences, then one easy way to prove that is Cauchy is to use the Cauchy criterion. Explain!
3. Give a direct argument that is Cauchy that does not use the Cauchy criterion.
4. Do the same for the product, .
5. Let be Cauchy sequences. Decide whether or not each of the following is Cauchy, justifying each conclusion.

8. where refers to the greatest integer less than or equal to x.
9. Consider the following (invented) definition: A sequence is ***pseudo-Cauchy*** if, for all | < Decide which one of the following two statements is True. Provide a counterexample for the other.
10. Pseudo-Cauchy sequences are bounded.
11. If and are pseudo-Cauchy, then is pseudo-Cauchy as well.

