Math 351: class discussion, 10 October 2018



Cauchy sequences & supremum, infimum of sets

1. Review: What is the ***Cauchy criterion*** for convergence?
2. Given a sequence $\left\{a\_{n}\right\} that has the property \left|a\_{n}-a\_{n+1}\right|\leq \frac{1}{2^{n }} .$ for all n. Must it follow that $\left\{a\_{n}\right\} $be Cauchy?
3. Show directly that $b\_{n}= \frac{1}{1!}+ \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+…+\frac{1}{n!} a Cauchy sequence.$

*Hint:* First note that $\frac{1}{n!} \leq \frac{1}{2^{n}} for all n\geq 4.$

1. Consider the following statement: “Given $any ϵ>0, a\_{n+1}\_{ ϵ}^{ ≈} a\_{n}$ for n >>1”

Give an *example* of an increasing sequence that satisfies this condition, yet is not a Cauchy sequence.

1. Given a sequence $\left\{a\_{j}\right\} that has the property \left|a\_{n}-a\_{k}\right|\leq \frac{1}{n+k}$ for all *n* and *k*. Prove that $\left\{a\_{n}\right\} $is Cauchy.
2. **Definitions:** Let $S⊆R.$
3. An ***upper bound*** for S is \_\_\_\_\_\_\_\_\_\_\_\_.
4. S is ***bounded above*** if \_\_\_\_\_\_\_\_\_\_\_\_.
5. The ***maximum*** of S is \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.
6. **Definition:** Let $S⊆R.$ The ***supremum*** of S (abbreviated sup S, aka lub S) is \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.
7. Let $S⊆R.$ Prove that:
8. If max S exists, then it is unique.
9. If sup S exists, then it is unique.
10. *[exercises from a graduate Finance program]*
11. State what it means for a sequence *not* to be Cauchy. Use quantifiers.
12. Prove that if $\left\{a\_{j}\right\} $is Cauchy then $\left\{a\_{j}^{2}\right\} $ is also Cauchy.
13. Give an example of a Cauchy sequence $\left\{a\_{j}^{2}\right\} $such that $\left\{a\_{j}\right\}$ is not Cauchy.
14. Prove the ***Completeness Property for sets***, *viz.*

If S $⊆R $is non-empty and bounded above, then sup S exists.

1. Introduce ***infimum*** of S (*aka* glb)
2. Is there any sequence of numbers a1, a2, . . . such that the set {a1, a2, . . .} is bounded, but the sequence has no maximal and no minimal elements?
3. Find the supremum for the following set and prove that your answer is correct. S = $\left\{\frac{1}{2}, \frac{2}{3},\frac{3}{4}, …,\frac{n}{n+1}\right\}$
4. Consider the set A = { (−1)n n : n ∈ N}. (a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
5. Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?
6. (Mattuck, Example 6.4) Consider the recursively defined sequence $\left\{a\_{j}\right\}, where a\_{1}=1 and a\_{n+1}=\frac{1}{a\_{n}+1} ∀n\geq 1$.

*Prove that this is a Cauchy sequence and determine its limit.*

1. Consider the set A = {x ∈ **R**: 1 < x < 2}.

(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?

(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

15. Prove that if S ⊂ R is non-empty and bounded below, then it has an infimum.

16. (UC, Berkeley) For S ⊂ R a non-empty subset that is bounded above and x ∈ R, let xS be the set {xs: s ∈ S}.

 (a) Show that if x > 0, then sup (xS) = x sup (S).

(b) Show that if x < 0, then inf (xS) = x inf (S).

17. (UC, Berkeley) Let S, T $⊆$ **R** be non-empty subsets that are bounded from above, and define

S + T = {s + t: s ∈ S, t ∈ T}.

Show sup(S + T) = sup(S) + sup(T). Then, use this to prove that if x ∈ R and S +x is the set {s+x: s ∈ S}, then

sup(S + x) = sup(S) + x.

Additional Exercises (S. Abbott, **Understanding Analysis**, 2nd edition, Springer)

1. Decide whether each of the following statements is True or False. Provide either a brief justification or a counterexample.
2. If every proper subsequence of $\left\{x\_{n}\right\}$ converges, then $\left\{x\_{n}\right\}$ converges as well.
3. If $\left\{a\_{n}\right\} contains a divergent subsequence, then \left\{a\_{n}\right\} diverges.$
4. If $\left\{a\_{n}\right\} is bounded and diverges, then there exist two subsequences of \left\{a\_{n}\right\} that converge$

to different limits.

1. If $\left\{a\_{n}\right\} is monotone and contains a convergent subsequence, then \left\{a\_{n}\right\} converges.$.
2. If $\left\{a\_{n}\right\} and \left\{b\_{n}\right\}$ are Cauchy sequences, then one easy way to prove that $\left\{a\_{n}+b\_{n}\right\}$ is Cauchy is to use the Cauchy criterion. Explain!
3. Give a direct argument that $\left\{a\_{n}+b\_{n}\right\} $is Cauchy that does not use the Cauchy criterion.
4. Do the same for the product, $\left\{a\_{n}b\_{n}\right\}$.
5. Let $\left\{a\_{n}\right\} and \left\{b\_{n}\right\}$ be Cauchy sequences. Decide whether or not each of the following is Cauchy, justifying each conclusion.
6. $c\_{n}=\left|a\_{n}-b\_{n}\right|$
7. $c\_{n}=(-1)^{n}a\_{n}$
8. $ c\_{n}=\left⟦a\_{n}\right⟧$ where $\left⟦x\right⟧$ refers to the greatest integer less than or equal to x.
9. Consider the following (invented) definition: A sequence $\left\{a\_{n}\right\} $is ***pseudo-Cauchy*** if, for all $ϵ>0, there exists an N such that if n\geq N, then |a\_{n+1}-a\_{n}$| < $ϵ.$ Decide which one of the following two statements is True. Provide a counterexample for the other.
10. Pseudo-Cauchy sequences are bounded.
11. If $\left\{a\_{n}\right\}$ and $\left\{b\_{n}\right\}$ are pseudo-Cauchy, then $\left\{a\_{n}+b\_{n}\right\} $is pseudo-Cauchy as well.

