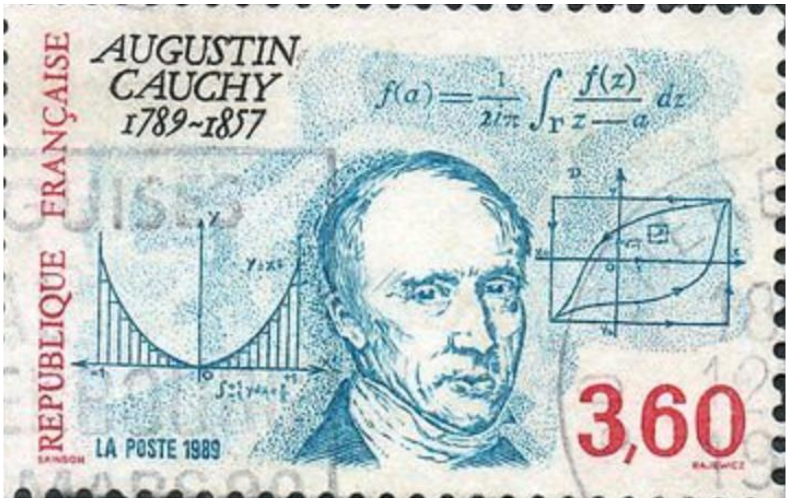
MATH 351: CLASS DISCUSSION, 10 OCTOBER 2018



Cauchy sequences & supremum, infimum of sets

1. Review: What is the ***Cauchy criterion*** for convergence?
2. Given a sequence $\{a_n\}$ *that has the property* $|a_n - a_{n+1}| \leq \frac{1}{2^n}$. for all n. Must it follow that $\{a_n\}$ be Cauchy?
3. Show directly that $b_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$ *a Cauchy sequence*.

   Hint: First note that $\frac{1}{n!} \leq \frac{1}{2^n}$ *for all* $n \geq 4$.
4. Consider the following statement: "Given *any* $\epsilon > 0$, $a_{n+1} \underset{\epsilon}{\approx} a_n$ for n >>1"

   Give an *example* of an increasing sequence that satisfies this condition, yet is not a Cauchy sequence.
5. Given a sequence $\{a_j\}$ *that has the property* $|a_n - a_k| \leq \frac{1}{n+k}$ for all *n* and *k*. Prove that $\{a_n\}$ is Cauchy.
6. **Definitions:** Let $S \subseteq \mathbb{R}$.
   (a) An ***upper bound*** for S is _____________.
   (b) S is ***bounded above*** if _____________.
   (c) The ***maximum*** of S is _________________.

7. **Definition:** Let $S \subseteq \mathbb{R}$. The ***supremum*** of S (abbreviated sup S, aka lub S) is ____________________.
8. Let $S \subseteq \mathbb{R}$. Prove that:
   (a) If max S exists, then it is unique.
   (b) If sup S exists, then it is unique.
9. *[exercises from a graduate Finance program]*
   (a) State what it means for a sequence *not* to be Cauchy. Use quantifiers.
   (b) Prove that if $\{a_j\}$ is Cauchy then $\{a_j{}^2\}$ is also Cauchy.
   (c) Give an example of a Cauchy sequence $\{a_j{}^2\}$ such that $\{a_j\}$ is not Cauchy.

10. Prove the ***Completeness Property for sets***, *viz.*

    If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then sup S exists.

11. Introduce ***infimum*** of S (*aka* glb)

12. Is there any sequence of numbers $a_1, a_2, \ldots$ such that the set $\{a_1, a_2, \ldots\}$ is bounded, but the sequence has no maximal and no minimal elements?

13. Find the supremum for the following set and prove that your answer is correct. $S = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}\right\}$

14. Consider the set $A = \{ (-1)^n n : n \in N\}$. (a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?

    (c) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

15. (Mattuck, Example 6.4) Consider the recursively defined sequence $\{a_j\}$, *where* $a_1 = 1$ *and* $a_{n+1} = \frac{1}{a_n+1}$ $\forall n \geq 1.$

    Prove that this is a Cauchy sequence and determine its limit.

16. Consider the set $A = \{x \in \mathbf{R}: 1 < x < 2\}$.

    (a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?

    (b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

15. Prove that if $S \subset R$ is non-empty and bounded below, then it has an infimum.

16. (UC, Berkeley) For $S \subset R$ a non-empty subset that is bounded above and $x \in R$, let xS be the set $\{xs: s \in S\}$.

    (a) Show that if $x > 0$, then $\sup(xS) = x \sup(S)$.

    (b) Show that if $x < 0$, then $\inf(xS) = x \inf(S)$.

17. (UC, Berkeley) Let $S, T \subseteq \mathbf{R}$ be non-empty subsets that are bounded from above, and define

$S + T = \{s + t: s \in S, t \in T\}$.

Show $\sup(S + T) = \sup(S) + \sup(T)$. Then, use this to prove that if $x \in R$ and $S + x$ is the set $\{s+x: s \in S\}$, then $\sup(S + x) = \sup(S) + x$.

Additional Exercises (S. Abbott, **Understanding Analysis**, 2nd edition, Springer)

1. Decide whether each of the following statements is True or False. Provide either a brief justification or a counterexample.

   (a) If every proper subsequence of $\{x_n\}$ converges, then $\{x_n\}$ converges as well.

   (b) If $\{a_n\}$ contains a divergent subsequence, then $\{a_n\}$ diverges.

   (c) If $\{a_n\}$ is bounded and diverges, then there exist two subsequences of $\{a_n\}$ that converge to different limits.

   (d) If $\{a_n\}$ is monotone and contains a convergent subsequence, then $\{a_n\}$ converges..

2. If $\{a_n\}$ *and* $\{b_n\}$ are Cauchy sequences, then one easy way to prove that $\{a_n + b_n\}$ is Cauchy is to use the Cauchy criterion. Explain!

   (a) Give a direct argument that $\{a_n + b_n\}$ is Cauchy that does not use the Cauchy criterion.

   (b) Do the same for the product, $\{a_n b_n\}$.

3. Let $\{a_n\}$ *and* $\{b_n\}$ be Cauchy sequences. Decide whether or not each of the following is Cauchy, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [\![a_n]\!]$ where $[\![x]\!]$ refers to the greatest integer less than or equal to x.

4. Consider the following (invented) definition: A sequence $\{a_n\}$ is ***pseudo-Cauchy*** if, for all $\epsilon > 0$, *there exists an N such that if* $n \geq N$, *then* $|a_{n+1} - a_n| < \epsilon$. Decide which one of the following two statements is True. Provide a counterexample for the other.

(a) Pseudo-Cauchy sequences are bounded.

(b) If $\{a_n\}$ and $\{b_n\}$ are pseudo-Cauchy, then $\{a_n + b_n\}$ is pseudo-Cauchy as well.