Math 351: class discussion, 12 October 2018

Completeness Theorem

lim sup; lim inf

1. Review: If are Cauchy sequences, then one easy way to prove that is Cauchy is to use the Cauchy criterion. Explain!
2. Give a direct argument that is Cauchy that does not use the Cauchy criterion.
3. Do the same for the product, .
4. **Definition:** Let non-empty. The ***supremum*** of S (abbreviated sup S, *aka* lub S) is \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.
5. Let non-empty. Prove that:
6. If max S exists, then it is unique.
7. If sup S exists, then it is unique.
8. *[exercises from a graduate Finance program]*
9. State what it means for a sequence *not* to be Cauchy. Use quantifiers.
10. Prove that if is Cauchy then is also Cauchy.
11. Give an example of a Cauchy sequence such that is not Cauchy.
12. Prove the ***Completeness Property for sets***, *viz.*

If S is non-empty and bounded above, then sup S exists.

1. Define ***infimum*** of S (*aka* glb)
2. Is there any sequence of numbers a1, a2, . . . such that the set {a1, a2, . . .} is bounded, but the sequence has no maximal and no minimal elements?
3. Find the supremum for the following set and prove that your answer is correct. S =
4. Consider the set A = { (−1)n n : n ∈ **N**}.
5. Show that A is bounded above. Find the supremum of A. Is this supremum a maximum of A?
6. Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?
7. (Mattuck, Example 6.4) Consider the recursively defined sequence .

*Prove that this is a Cauchy sequence and determine its limit.*

1. Consider the set A = {x ∈ **R**: 1 < x < 2}.

(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?

(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

12. . Prove that if S ⊂ R is non-empty and bounded below, then it has an infimum.

13. (UC, Berkeley) For S ⊂ R a non-empty subset that is bounded above and x ∈ R, let xS be the set {xs: s ∈ S}.

(a) Show that if x > 0, then sup (xS) = x sup (S).

(b) Show that if x < 0, then inf (xS) = x inf (S).

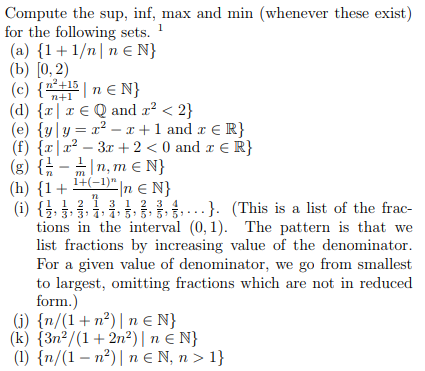
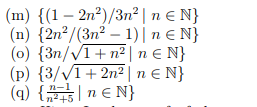
14. (UC, Berkeley) Let S, T **R** be non-empty subsets that are bounded from above, and define

S + T = {s + t: s ∈ S, t ∈ T}.

Show sup(S + T) = sup(S) + sup(T). Then, use this to prove that if x ∈ R and S +x is the set {s+x: s ∈ S}, then

sup(S + x) = sup(S) + x.

1. Find the max, min, sup, and inf of the following and justify each answer.
2. Find the max, min, sup, and inf of the following and justify each answer.
3. *(Purdue)*

(s)

1. (Mattuck; 6.5.3) Assume that A and B are bounded non-empty subsets of R. Prove the following (for each part you can assume any of the preceding parts).

If c is a constant then we define cA = {ca | a}.

Also, we define A + B = {a + b: a ∈ A, b ∈ B}.

1. 𝐴
2. If c > 0, then sup cA = c sup A
3. If c > 0, then inf ca = c inf A
4. sup(-A) = - inf A
5. inf (-A) = - sup A
6. sup(A + B)
7. inf(A + B) inf A + inf B
8. Let {an} be a bounded sequence. Define lim sup an and lim inf an.
9. For each of the following sequences

12. (challenge problem) Let for a fixed value of t > 0.
13. Determine for each of the following sequences:

Let . Determine

1. Prove that lim sup ( ≤ lim sup (. Give an example when equality does not hold.
2. Prove that lim inf ≤ lim inf ( Give an example when equality does not hold.
3. State and prove results similar to those of problem 14 for the product of two sequences with *positive* terms.
4. Let and let

. Find:

(b) lim inf ( + )

(c)

(d) lim sup ( + )

(e) lim sup + lim sup

1. Prove the Proposition: If lim sup