MATH 351: CLASS DISCUSSION, 12 OCTOBER 2018

Completeness Theorem

lim sup; lim inf

1. Review: If $\{a_n\}$ *and* $\{b_n\}$ are Cauchy sequences, then one easy way to prove that $\{a_n + b_n\}$ is Cauchy is to use the Cauchy criterion. Explain!
   (a) Give a direct argument that $\{a_n + b_n\}$ is Cauchy that does not use the Cauchy criterion.
   (b) Do the same for the product, $\{a_n b_n\}$.
2. **Definition:** Let $S \subseteq \mathbb{R}$ *be* non-empty. The ***supremum*** of S (abbreviated sup S, *aka* lub S) is ____________________.
3. Let $S \subseteq \mathbb{R}$ *be* non-empty. Prove that:
   (a) If max S exists, then it is unique.
   (b) If sup S exists, then it is unique.
4. *[exercises from a graduate Finance program]*
   (a) State what it means for a sequence *not* to be Cauchy. Use quantifiers.
   (b) Prove that if $\{a_j\}$ is Cauchy then $\{a_j{}^2\}$ is also Cauchy.
   (c) Give an example of a Cauchy sequence $\{a_j{}^2\}$ such that $\{a_j\}$ is not Cauchy.
5. Prove the ***Completeness Property for sets***, *viz.*
   If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then sup S exists.
6. Define ***infimum*** of S (*aka* glb)
7. Is there any sequence of numbers $a_1, a_2, \ldots$ such that the set $\{a_1, a_2, \ldots\}$ is bounded, but the sequence has no maximal and no minimal elements?
8. Find the supremum for the following set and prove that your answer is correct. S = $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}\right\}$
9. Consider the set A = $\{ (-1)^n n : n \in \mathbf{N}\}$.
   (a) Show that A is bounded above. Find the supremum of A. Is this supremum a maximum of A?
   (b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?
10. (Mattuck, Example 6.4) Consider the recursively defined sequence $\{a_j\}$, *where* $a_1 = 1$ *and* $a_{n+1} = \frac{1}{a_n+1}$ $\forall n \geq 1$. *Prove that this is a Cauchy sequence and determine its limit.*
11. Consider the set A = $\{x \in \mathbf{R}: 1 < x < 2\}$.
    (a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
    (b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?
12. . Prove that if $S \subset R$ is non-empty and bounded below, then it has an infimum.
13. (UC, Berkeley) For $S \subset R$ a non-empty subset that is bounded above and $x \in R$, let xS be the set $\{xs: s \in S\}$.
    (a) Show that if $x > 0$, then sup (xS) = x sup (S).
    (b) Show that if $x < 0$, then inf (xS) = x inf (S).

14. (UC, Berkeley) Let S, T $\subseteq$ **R** be non-empty subsets that are bounded from above, and define

S + T = {s + t: s ∈ S, t ∈ T}.

Show sup(S + T) = sup(S) + sup(T). Then, use this to prove that if x ∈ R and S +x is the set {s+x: s ∈ S}, then sup(S + x) = sup(S) + x.

15. Find the max, min, sup, and inf of the following and justify each answer.

$$S = \left\{\frac{2n+1}{n+1} : n \geq 1\right\}$$

16. Find the max, min, sup, and inf of the following and justify each answer.

$$T = \left\{\frac{n+(-1)^n n}{n+1} : n \geq 1\right\}$$

17. (Purdue)

Compute the sup, inf, max and min (whenever these exist) for the following sets. [1]

(a) $\{1+1/n \mid n \in \mathbb{N}\}$
(b) $[0,2)$
(c) $\{\frac{n^2+15}{n+1} \mid n \in \mathbb{N}\}$
(d) $\{x \mid x \in \mathbb{Q} \text{ and } x^2 < 2\}$
(e) $\{y \mid y = x^2 - x + 1 \text{ and } x \in \mathbb{R}\}$
(f) $\{x \mid x^2 - 3x + 2 < 0 \text{ and } x \in \mathbb{R}\}$
(g) $\{\frac{1}{n} - \frac{1}{m} \mid n, m \in \mathbb{N}\}$
(h) $\{1 + \frac{1+(-1)^n}{n} \mid n \in \mathbb{N}\}$
(i) $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\}$. (This is a list of the fractions in the interval $(0,1)$. The pattern is that we list fractions by increasing value of the denominator. For a given value of denominator, we go from smallest to largest, omitting fractions which are not in reduced form.)
(j) $\{n/(1+n^2) \mid n \in \mathbb{N}\}$
(k) $\{3n^2/(1+2n^2) \mid n \in \mathbb{N}\}$
(l) $\{n/(1-n^2) \mid n \in \mathbb{N},\ n > 1\}$
(m) $\{(1-2n^2)/3n^2 \mid n \in \mathbb{N}\}$
(n) $\{2n^2/(3n^2-1) \mid n \in \mathbb{N}\}$
(o) $\{3n/\sqrt{1+n^2} \mid n \in \mathbb{N}\}$
(p) $\{3/\sqrt{1+2n^2} \mid n \in \mathbb{N}\}$
(q) $\{\frac{n-1}{n^2+5} \mid n \in \mathbb{N}\}$
(r) $\{\frac{3n+1}{n+1} \mid n \in \mathbb{N}\}$
(s) $\{n^{(-1)^n} \mid n \geq 1\}$

18. (Mattuck; 6.5.3) Assume that A and B are bounded non-empty subsets of R. Prove the following (for each part you can assume any of the preceding parts).

If c is a constant then we define cA = {ca | a∈ A}.

Also, we define A + B = {a + b: a ∈ A, b ∈ B}.

(a) $A \subseteq B \Rightarrow \sup(A) \leq \sup(B)$
(b) $A \subseteq B \Rightarrow \inf(A) \geq \inf(B)$
(c) If c > 0, then sup cA = c sup A
(d) If c > 0, then inf ca = c inf A
(e) sup(-A) = - inf A
(f) inf (-A) = - sup A

(g) $\sup(A + B) \leq \sup A + \sup B$

(h) $\inf(A + B) \geq \inf A + \inf B$

19. Let $\{a_n\}$ be a bounded sequence. Define lim sup a_n and lim inf a_n.

20. For each of the following sequences $\{a_n\}$, *detemine* $\limsup a_n$ *and* $\liminf a_n$.

(a) $a_n = (-1)^n$

(b) $a_n = \cos\frac{n\pi}{2}$

(c) $a_n = (arc\tan n)\sin\frac{n\pi}{2}$

21. (challenge problem) Let $a_n = \cos\sqrt{t + n^2\pi^2}$ for a fixed value of t > 0. *Detemine* $\limsup a_n$ *and* $\liminf a_n$.

22. Determine $\limsup a_n$ *and* $\liminf a_n$ for each of the following sequences:

Let $a_n = \frac{1+(-1)^n 2n}{1+3n}$. Determine $\limsup a_n$ *and* $\liminf a_n$.

23. Prove that $\limsup (a_n + b_n) \leq \limsup (a_n + b_n)$. Give an example when equality does not hold.

24. Prove that $\liminf a_n + \liminf b_n \leq \liminf (a_n + b_n)$ Give an example when equality does not hold.

25. State and prove results similar to those of problem 14 for the product $\{a_n b_n\}$ of two sequences with *positive* terms.

26. Let $\{a_n\}$ *be the sequence* $0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1$ .... and let $\{b_n\}$ *be the sequence* $2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0$ ... . Find:

(a) $\liminf a_n + \liminf b_n$

(b) $\liminf (a_n + b_n)$

(c) $\liminf a_n + \limsup b_n$

(d) $\limsup (a_n + b_n)$

(e) $\limsup a_n + \limsup b_n$

27. Prove the Proposition: If $\limsup a_n = s^*$, *then there exists a subsequence of* $\{a_n\}$ *that converges to* s^*.