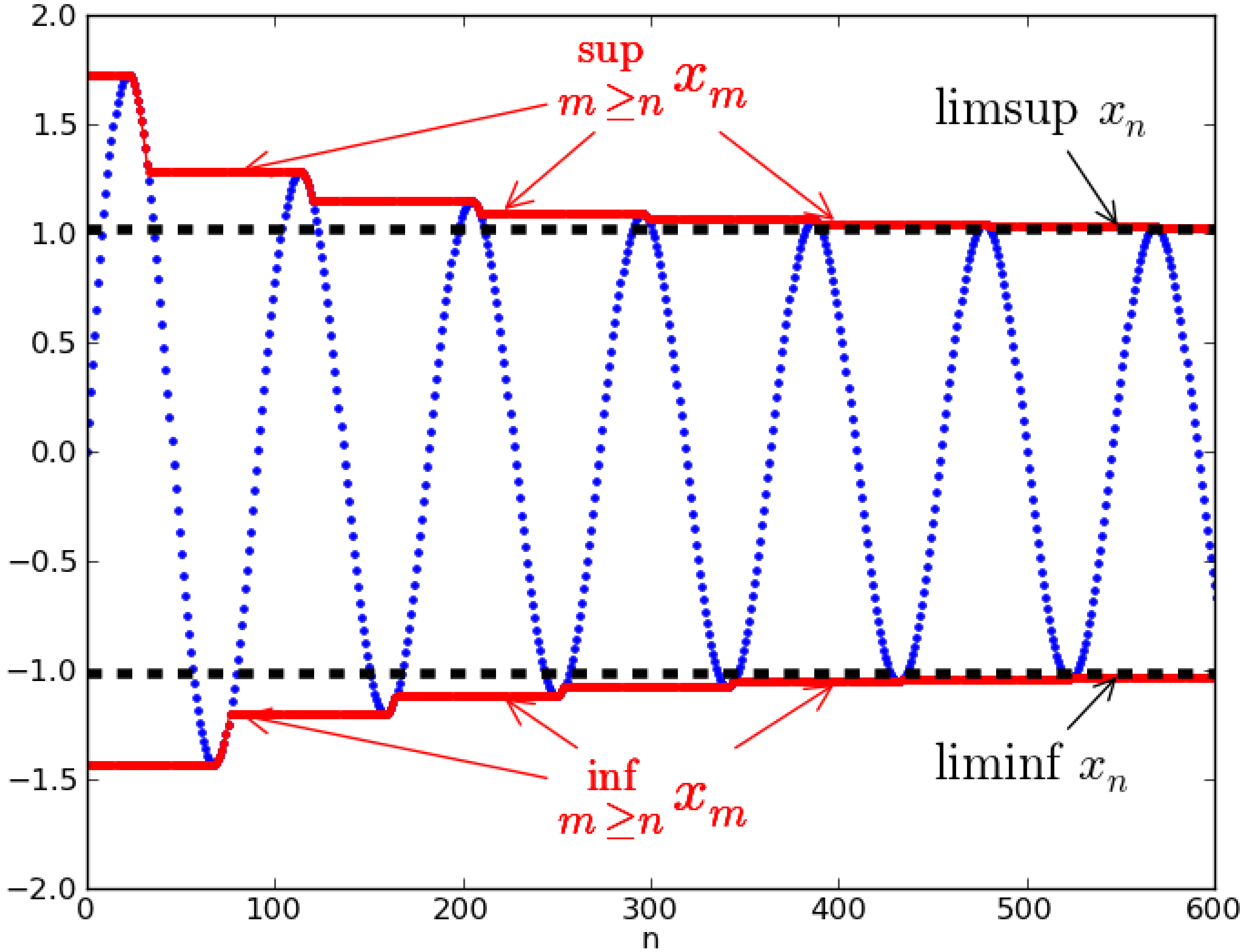
MATH 351: CLASS DISCUSSION, 15 OCTOBER 2018

lim sup; lim inf; intro to series

Review:



An illustration of limit superior and limit inferior. The sequence x_n is shown in blue. The two red curves approach the limit superior and limit inferior of x_n, shown as dashed black lines. In this case, the sequence accumulates around the two limits. The superior limit is the larger of the two, and the inferior limit is the smaller of the two. The inferior and superior limits agree if and only if the sequence is convergent (i.e., when there is a single limit). (Wikipedia)

1. *(Purdue)*

Compute the sup, inf, max and min (whenever these exist) for the following sets. [1]

(a) $\{1 + 1/n \mid n \in \mathbb{N}\}$
(b) $[0, 2)$
(c) $\{\frac{n^2+15}{n+1} \mid n \in \mathbb{N}\}$
(d) $\{x \mid x \in \mathbb{Q} \text{ and } x^2 < 2\}$
(e) $\{y \mid y = x^2 - x + 1 \text{ and } x \in \mathbb{R}\}$
(f) $\{x \mid x^2 - 3x + 2 < 0 \text{ and } x \in \mathbb{R}\}$
(g) $\{\frac{1}{n} - \frac{1}{m} \mid n, m \in \mathbb{N}\}$
(h) $\{1 + \frac{1+(-1)^n}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots\}$. (This is a list of the fractions in the interval $(0,1)$. The pattern is that we list fractions by increasing value of the denominator. For a given value of denominator, we go from smallest to largest, omitting fractions which are not in reduced form.)

(j) $\{n/(1+n^2) \mid n \in \mathbb{N}\}$

(k) $\{3n^2/(1+2n^2) \mid n \in \mathbb{N}\}$

(l) $\{n/(1-n^2) \mid n \in \mathbb{N},\, n > 1\}$

(m) $\{(1-2n^2)/3n^2 \mid n \in \mathbb{N}\}$

(n) $\{2n^2/(3n^2-1) \mid n \in \mathbb{N}\}$

(o) $\{3n/\sqrt{1+n^2} \mid n \in \mathbb{N}\}$

(p) $\{3/\sqrt{1+2n^2} \mid n \in \mathbb{N}\}$

(q) $\{\frac{n-1}{n^2+5} \mid n \in \mathbb{N}\}$

(r) $\{\frac{3n+1}{n+1} \mid n \in \mathbb{N}\}$

(s) $\{n^{(-1)^n} \mid n \geq 1\}$

2. (Mattuck; 6.5.3) Assume that A and B are bounded non-empty subsets of R. Prove the following (for each part you can assume any of the preceding parts).

If c is a constant then we define cA = {ca | a$\in A$}.

Also, we define A + B = {a + b: a ∈ A, b ∈ B}.

(a) $A \subseteq B \Rightarrow \sup(A) \leq \sup(B)$

(b) $A \subseteq B \Rightarrow \inf(A) \geq \inf(B)$

(c) If c > 0, then sup cA = c sup A

(d) If c > 0, then inf ca = c inf A

(e) sup(-A) = - inf A

(f) inf (-A) = - sup A

(g) sup(A + B) ≤ sup A + sup B

(h) inf(A + B) ≥ inf A + inf B

3. Let $\{a_n\}$ be a bounded sequence, recall the definitions of *lim sup* a_n and *lim inf* a_n.

4. For each of the following sequences $\{a_n\}$, *detemine* $\limsup a_n$ *and* $\liminf a_n$.

(a) $a_n = (-1)^n$

(b) $a_n = \cos\frac{n\pi}{2}$

(c) $a_n = (arc\tan n)\sin\frac{n\pi}{2}$

(d) $a_n = \frac{1+(-1)^n 2n}{1+3n}$.

5. Let $a_n = (-1)^n + \frac{1}{n} + 3\sin\frac{\pi n}{2}$. Let A = $\{a_n \mid n \geq 1\}$.

Find sup A, inf A, lim sup a_n and lim inf a_n.

6. Define the sequence $\{a_n\}$ *as follows*:

$$a_n = \begin{cases} 2^{\frac{1}{n+1}} & if\ n\ is\ odd \\ 1 & if\ n\ is\ even \end{cases}$$

Calculate lim sup a_n and lim inf a_n.

7. Let $\{a_n\}$ *be the sequence* $0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1$ .... and let $\{b_n\}$ *be the sequence* $2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0$ ... . Find:

(a) $\liminf a_n + \liminf b_n$

(b) lim inf (a_n + b_n)

(c) $\liminf a_n + \limsup b_n$

(d) lim sup (a_n + b_n)

(e) lim sup a_n + lim sup b_n

8. Prove that $\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$ Give an example when equality does not hold.

9. Prove that $\liminf a_n + \liminf b_n \leq \liminf (a_n + b_n)$ Give an example when equality does not hold.

10. Prove the Proposition: If $\limsup a_n = s^*, \textit{then there exists a subsequence of } \{a_n\} \textit{ that converges to } s^*$.

Numerical series

11. Define: infinite series; sequence of partial sums; convergent series; divergent series; sum of a series

12. Review the series (a) $\sum_{n=1}^{\infty} \frac{1}{n}$ and (b) $\sum_{n=0}^{\infty} r^n$ for |r| < 1

13. How can a sequence be converted into a series?

14. State and prove **the n^th^ term test for divergence** of series.

15. State the **Tail-convergence theorem** for series.

16. State and prove the **Linearity theorem** for series.

17. State and prove the **Comparison theorem** for positive series.

18. State and prove the **Ratio test** for series.

19. State and prove the **Integral test** for positive series.

20. State and prove the n^th^ root test for series.

MIT 18.01 (calculus) exercises on series:

7A-1 For each of the following series, determine convergence or divergence? Justify your answer. If the series converges, find its sum.

a) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{4^n} + \ldots$

b) $1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$

c) $1 + \frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} + \ldots$

d) $\ln 2 + \ln \sqrt{2} + \ln \sqrt[3]{2} + \ln \sqrt[4]{2} + \ldots$

e) $\sum_{1}^{\infty} \frac{2^{n-1}}{3^n}$

f) $\sum_{0}^{\infty} (-1)^n \frac{1}{3^n}$

7A-2 Find the rational number represented by the infinite decimal .21111 . . . .

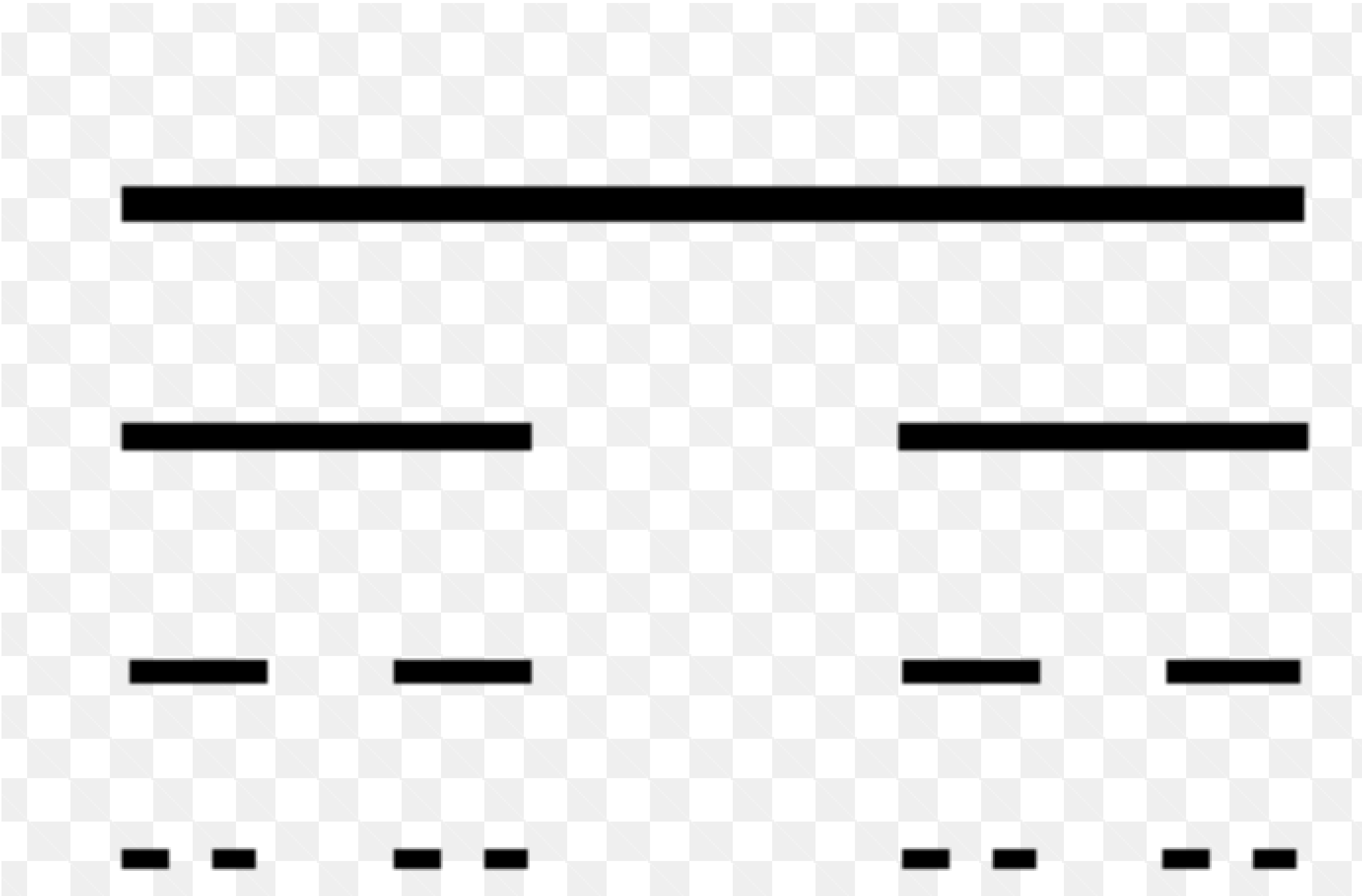
7A-3 For which x does the series $\sum_{0}^{\infty} \left(\frac{x}{2}\right)^n$ converge? For these values, find its sum $f(x)$.

7A-4 Find the sum of these series by first finding the partial sum S_n.

a) $\sum_{1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

b) $\sum_{1}^{\infty} \frac{1}{n(n+2)}$. (Hint: $\frac{1}{n(n+2)} = \frac{a}{n} + \frac{b}{n+2}$ for suitable a, b).

7A-5 A ball is dropped from height h; each time it lands, it bounces back 2/3 of the height from which it previously fell. What is the total distance (up and down) the ball travels?



Constructing the Cantor set