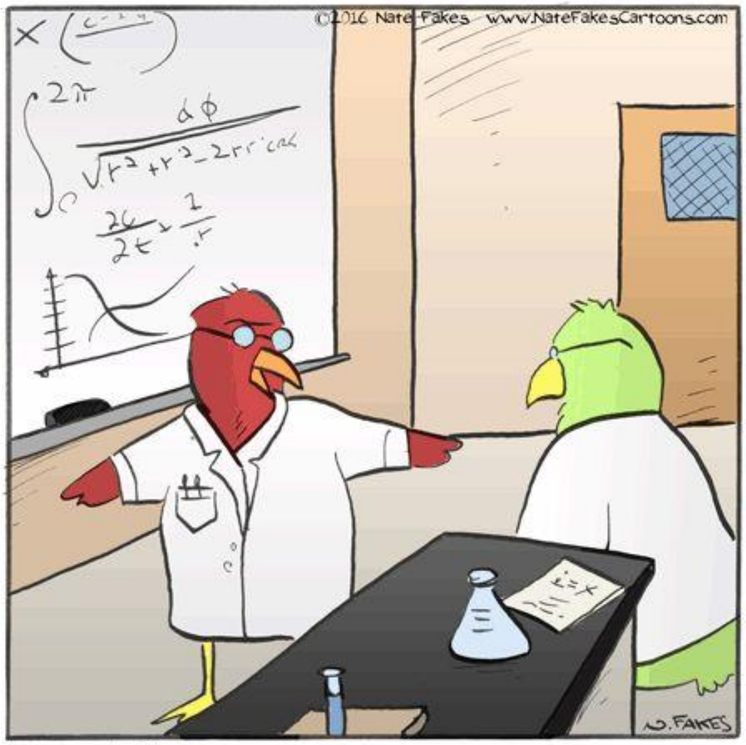
MATH 351: CLASS DISCUSSION, 24 OCTOBER 2018



"Well, I can't figure it out either, Petey. I wish we weren't bird brains!"

SERIES: RATIO TEST, ROOT TEST, ASYMPTOTIC CONVERGENCE TEST, ABSOLUTE CONVERGENCE, CAUCHY'S THEOREM, REARRANGEMENTS

Continuation: Numerical series

1. Review the series (a) $\sum_{n=1}^{\infty} \frac{1}{n}$ and (b) $\sum_{n=0}^{\infty} r^n$ for $|r| < 1$

2. What is meant by a "telescoping series"?

3. What is the **Cantor set**?

4. **Review:** Define conditional convergence; absolute convergence; alternating series

5. **Review:** **Prove** that absolute convergence implies convergence. Is the converse true?

6. State and prove the **Ratio test** for series.

7. State and prove the **Integral test** for positive series.

8. State and prove the **n^{th} root test** for series.

9. State and prove the **Asymptotic Comparison Test.**

10. State and prove **Cauchy's test** for conditional convergence of an alternating series.

11. What is a **rearrangement** of a series?

12. State and prove the **rearrangement theorems**.

7A-1 For each of the following series, determine convergence or divergence? Justify your answer. If the series converges, find its sum.

a) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{4^n} + \ldots$

b) $1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$

c) $1 + \frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} + \ldots$

d) $\ln 2 + \ln \sqrt{2} + \ln \sqrt[3]{2} + \ln \sqrt[4]{2} + \ldots$

e) $\sum_{1}^{\infty} \frac{2^{n-1}}{3^n}$

f) $\sum_{0}^{\infty} (-1)^n \frac{1}{3^n}$

7A-2 Find the rational number represented by the infinite decimal .21111 . . . .

7A-3 For which x does the series $\sum_{0}^{\infty} \left(\frac{x}{2}\right)^n$ converge? For these values, find its sum $f(x)$.

7A-4 Find the sum of these series by first finding the partial sum S_n.

a) $\sum_{1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

b) $\sum_{1}^{\infty} \frac{1}{n(n+2)}$. (Hint: $\frac{1}{n(n+2)} = \frac{a}{n} + \frac{b}{n+2}$ for suitable a, b).

7A-5 A ball is dropped from height h; each time it lands, it bounces back 2/3 of the height from which it previously fell. What is the total distance (up and down) the ball travels?

7B: Convergence Tests

7B-1 Using the integral test, tell whether the following series converge or diverge; show work or reasoning.

a) $\sum_{0}^{\infty} \frac{n}{n^2+4}$ b) $\sum_{0}^{\infty} \frac{1}{n^2+1}$ c) $\sum_{0}^{\infty} \frac{1}{\sqrt{n+1}}$

d) $\sum_{1}^{\infty} \frac{\ln n}{n}$ e) $\sum_{2}^{\infty} \frac{1}{(\ln n)^p \cdot n}$ f) $\sum_{1}^{\infty} \frac{1}{n^p}$

(In the last two, the answer depends on the value of the parameter p.)

7B-2 Using the limit comparison test, tell whether each series converges or diverges; show work or reasoning. (For some of them, simple comparison works.)

a) $\sum_{1}^{\infty} \frac{1}{n^2+3n}$ b) $\sum_{1}^{\infty} \frac{1}{n+\sqrt{n}}$ c) $\sum_{1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

d) $\sum_{1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ e) $\sum_{1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ f) $\sum_{1}^{\infty} \frac{\ln n}{n}$

g) $\sum_{2}^{\infty} \frac{n^2}{n^4-1}$ h) $\sum_{1}^{\infty} \frac{n^3}{4n^4+n^2}$

7B-3 Prove that if $a_n > 0$ and $\sum_0^\infty a_n$ converges, then $\sum_0^\infty \sin a_n$ also converges.

7B-4 Using the ratio test, or otherwise, determine whether or not each of these series is absolutely convergent. (Note that $0! = 1$.)

a) $\displaystyle\sum_0^\infty \frac{n}{2^n}$ b) $\displaystyle\sum_0^\infty \frac{2^n}{n!}$ c) $\displaystyle\sum_1^\infty \frac{2^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

d) $\displaystyle\sum_0^\infty \frac{(n!)^2}{(2n)!}$ e) $\displaystyle\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ f) $\displaystyle\sum_1^\infty \frac{n!}{n^n}$; use $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

g) $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2}$ h) $\displaystyle\sum_0^\infty \frac{(-1)^n}{\sqrt{n^2+1}}$ i) $\displaystyle\sum_0^\infty \frac{n}{n+1}$

7B-5 For those series in **7B-4** which are *not* absolutely convergent, tell whether they are conditionally convergent or divergent.

Challenge problems: Stewart, Calculus

1.

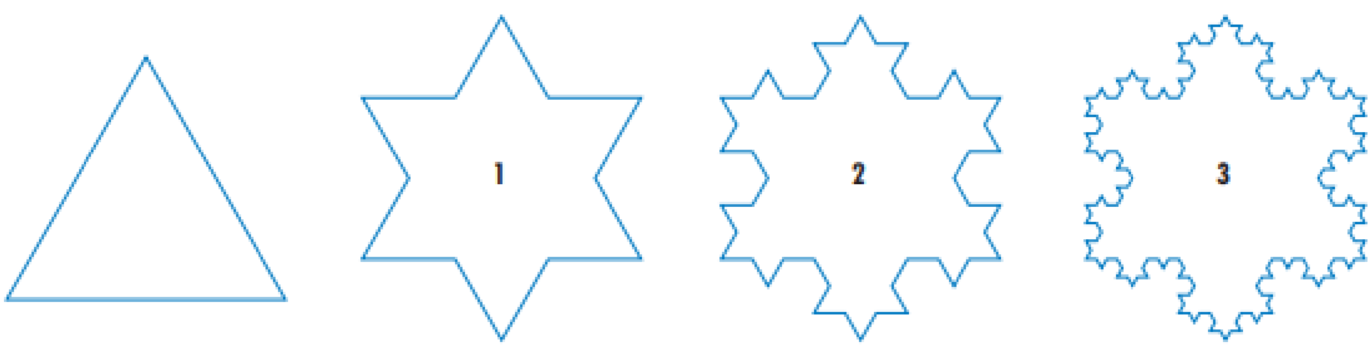
To construct the **snowflake curve**, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part (see the figure). Step 2 is to repeat Step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.

(a) Let s_n, l_n, and p_n represent the number of sides, the length of a side, and the total length of the nth approximating curve (the curve obtained after Step n of the construction), respectively. Find formulas for s_n, l_n, and p_n.

(b) Show that $p_n \to \infty$ as $n \to \infty$.

(c) Sum an infinite series to find the area enclosed by the snowflake curve.

Parts (b) and (c) show that the snowflake curve is infinitely long but encloses only a finite area.



2.

Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

3. Sum the series

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right).$$

4. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1 \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$

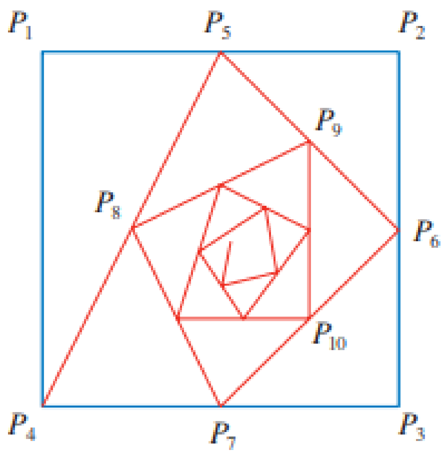
Find the sum of the series $\sum_{n=0}^{\infty} a_n$.

5.

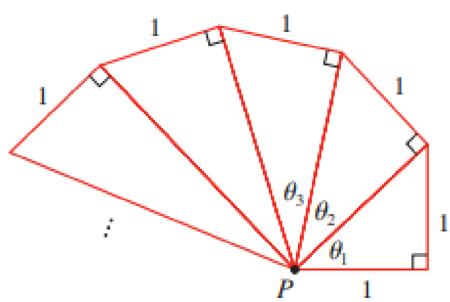
Starting with the vertices $P_1(0, 1)$, $P_2(1, 1)$, $P_3(1, 0)$, $P_4(0, 0)$ of a square, we construct further points as shown in the figure: P_5 is the midpoint of P_1P_2, P_6 is the midpoint of P_2P_3, P_7 is the midpoint of P_3P_4, and so on. The polygonal spiral path $P_1P_2P_3P_4P_5P_6P_7 \ldots$ approaches a point P inside the square.

(a) If the coordinates of P_n are (x_n, y_n), show that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$ and find a similar equation for the y-coordinates.

(b) Find the coordinates of P.

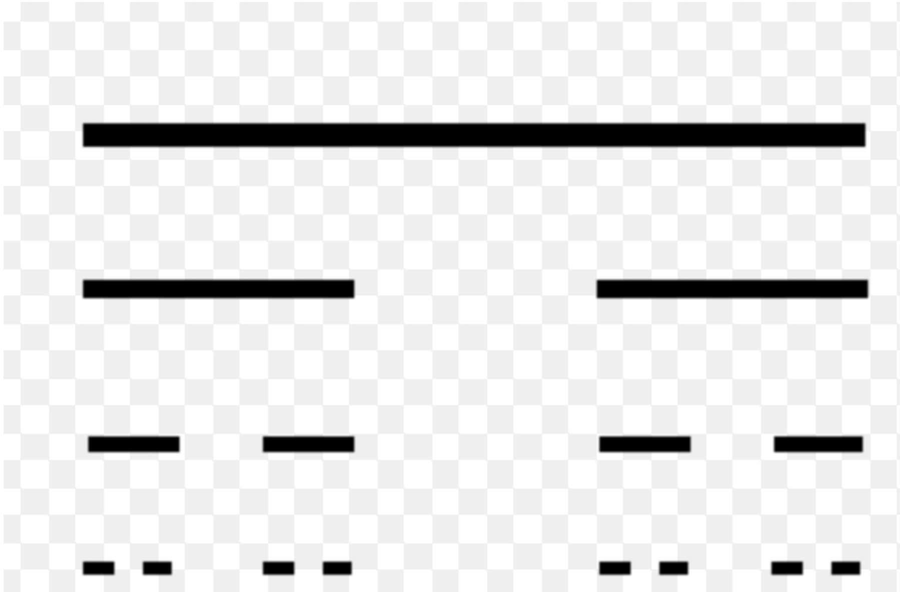


6. Right-angled triangles are constructed as in the figure. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes indefinitely many turns around P by showing that $\Sigma\, \theta_n$ is a divergent series.



The notion of infinity is our greatest friend; it is also the greatest enemy of our peace of mind.

– James Pierpont



Constructing the Cantor set