MATH 351: QUESTIONS FOR CLASS DISCUSSION, 26TH OCTOBER

BRIEF LOOK AT POWER SERIES

FUNCTIONS OF ONE VARIABLE: CONTINUITY

Review:

Define conditional convergence; absolute convergence; alternating series; rearrangement

State and prove Cauchy's test for conditional convergence of an alternating series.

Prove that absolute convergence implies convergence. Is the converse true?

State and prove the rearrangement theorems.

I Define: *radius of convergence*

1. Prove the theorem: For each power series $\sum a_n x^n$ there is a unique $R \geq 0$ such that $\sum a_n x^n$ converges absolutely for |x| < R and diverges for |x| > R.

2. Find the radius of convergence (and the sum) of each of the following power series:

(a) $\sum_{n=0}^{\infty} x^n, \quad |x| < 1$

(b) $\sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1$

(c) $\sum_{n=1}^{\infty} nx^n, \quad |x| < 1$

(d) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(e) $\sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1$

(f) $\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$

(g) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(h)

$$\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$$

(i)

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{5^{n-1}}.$$

(Smith College exercises)

Exercise 5.7. Use the ratio test to find the interval of convergence for each of the following power series.

(a) $\sum_{i=1}^{\infty} \frac{x^i}{i}$ (b) $\sum_{i=1}^{\infty} \frac{x^i}{2^i}$ (c) $\sum_{i=1}^{\infty} \frac{x^i}{i!}$

(d) $\sum_{i=1}^{\infty} ix^i$ (e) $\sum_{i=1}^{\infty} \frac{i^2}{2^i}x^i$ (f) $\sum_{i=1}^{\infty} i^{3i}$

(g) $\sum_{i=1}^{\infty} (1+x)^i$ (h) $\sum_{n=1}^{\infty} \frac{99}{n^n}x^n$ (i) $\sum_{n=1}^{\infty} n!x^n$

Exercise 5.11. Use the ratio test to find the interval of convergence for each of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ (c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

(d) $\sum_{n=1}^{\infty} nx^n$ (e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}x^n$ (f) $\sum_{n=1}^{\infty} \frac{n^3}{x^n}$

(g) $\sum_{n=1}^{\infty} (1+x)^n$ (h) $\sum_{n=1}^{\infty} \frac{9^n}{n^9}x^n$ (i) $\sum_{n=1}^{\infty} n!x^n$

II CONTINUITY & LIMITS

1. Define *function, domain, graph.*

Let f(x) be defined on (a, b) and let p ∈ (a, b). Define: **f(x) is continuous at x = p.**

Give both the "Mattuck" definition and the ϵ, δ *definitions*.

Define: f(x) is **continuous** on (a, b).

2. Prove, using only the definition of continuity, that each of the following functions is continuous on the given interval.
   (a) f(x) = x^2 on (-∞, ∞)
   (b) f(x) = 1/x on (0, ∞)
   (c) f(x) = $2x^3 - 4x$ on (-∞, ∞)
   (d) f(x) = $\frac{x}{3+x^2}$ on (-∞, ∞)

3. Prove that g(x) = sin x is continuous on (-∞, ∞). Hint: show that |sin a – sin b| ≤ |a – b| .

4. Using (3) prove that $v(x) = \int_0^{\pi} \frac{\sin xt}{t}\, dt$ *is continuous everywhere.*

5. What are the four types of discontinuities?

6. Define: right-continuity, left-continuity. Define: **f(x) is continuous on [a, b].**

7. Using (3) prove that $G(x) = \int_0^{\pi} \frac{\sin xt}{t}\, dt$ *is continuous everywhere.* Which fact(s) about the Riemann integral are you taking for granted?

8. What are the four types of *discontinuities*? For each of the four, provide an example.

9. Let f be defined for x near p. Define: **the limit of f(x) as $x \to p$ equals L.**

10. What is the relationship between continuity and limit? Define: limit as $x \to \infty$ *or* $x \to -\infty$

11. Prove each of the following results, using only the definition of limit.

(a) $\lim_{x\to 0} x \sin\frac{1}{x}$

(b) $\lim_{x\to 3+} \frac{|x^2-9|}{x-3}$

(c) $\lim_{x\to\infty} \frac{1}{3+x^2}$

(d) $\lim_{x\to 1} \frac{x^3-125}{x-5}$