MATH 351: CLASS DISCUSSION, 3 OCTOBER 2018

1. Complete the proof of the Nested Intervals Theorem.

2. Show that, for any $\alpha \in \mathbb{R}$, *there exists a sequence of nested intervals having intersection* $\{\alpha\}$.

3. Example. Let $a_0 = 0$, *and for* $n \geq 1$, *let* $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$

Using the nested interval theorem, prove that $\{a_n\}$ *converges*.

4. Definition: K is a ***cluster point*** of $\{a_n\}$ means that, for all $\varepsilon > 0$, there exist infinitely many *n* for which $|a_n - K| < \varepsilon$.

5. Find any (and all) cluster points for each of the following sequences:

(a) ½, 1/3, ¼, 1/5, …
(b) 1, 2, 3, 4, 5, …
(c) 1, 0, 1, 0, 1, 0, …
(d) 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, …
(e) 2, 3, 5, 7, 11, 13, …

6. Prove the Cluster Point Theorem: A sequence $\{a_n\}$ has a cluster point, K, if and only if there exists a subsequence of $\{a_n\}$ that converges to K.

7. Prove the Bolzano-Weierstrass Theorem: A bounded sequence $\{a_n\}$ *has a convegent subsequence*.

Exercises:

1. Give an example of each of the following or argue that no such example exists:
   (a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
   (b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these two values.
   (c) A sequence that contains subsequences converging to every point in the infinite set {1/2, 1/3, ¼, 1/5, …}.
   (d) A sequence that contains subsequences converging to every point in the infinite set {1, 1/2, 1/3, ¼, 1/5, …}, and no subsequences converging to points outside of this set.

2. Decide whether each of the following statements is True or False. Provide either a brief justification or a counterexample.
   (a) If every proper subsequence of $\{x_n\}$ converges, then $\{x_n\}$ converges as well.
   (b) If $\{a_n\}$ contains a divergent subsequence, then $\{a_n\}$ diverges.
   (c) If $\{a_n\}$ is bounded and diverges, then there exist two subsequences of $\{a_n\}$ that converge to different limits.
   (d) If $\{a_n\}$ is monotone and contains a convergent subsequence, then $\{a_n\}$ converges.

ε-Red Riding Hood and the Big Bad Bolzano-Weierstrass Theorem

Once upon a time[1] a long long time ago back when Fermat's Last Theorem would still fit in a margin, $\exists$ a little[2] girl named ε-Red Riding Hood (see Figure 1.1). ε-Red

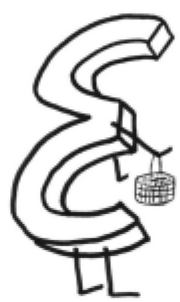

Figure 1.1: ε-Red Riding Hood with her basket of lemmas and π.

Riding Hood was trying to find the shortest path through the forest $\mathbb{F}$, a subfield of $\mathbb{X}$, to Γ's domain. She was carrying a basket full of lemmas[3] and π, to give to Γ (see Figure 1.2) who had a degenerate case of discontinuity.

Meanwhile, independently, the Big Bad Bolzano-Weierstrass Theorem (see Figure 1.3) was on a random walk through $\mathbb{F}$. As t approached T_0, T-time, the paths of ε-Red Riding Hood and the Big Bad Bolzano-Weierstrass Theorem converged.

"Hello ε-Red Riding Hood, may I ask you a question?", asked the Big Bad Bolzano-Weierstrass Theorem.

"You may indeed provided it is well-posed", stated ε-Red Riding Hood.

https://people.maths.ox.ac.uk/macdonald/errh/101_analysis_bedtime_stories_(epsilon_red_riding_hood).pdf