Math 351: class discussion, 5 October 2018

1. Complete the proof of the Bolzano-Weierstrass Theorem:

 A *bounded* sequence $\left\{a\_{n}\right\} has a convegent subsequence.$

1. Define Cauchy sequence.
2. How are Cauchy sequences related to convergent sequences? What is meant by “Cauchy criterion for convergence”?
3. Show directly that $b\_{n}= \frac{1}{1!}+ \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+…+\frac{1}{n!} a Cauchy sequence.$

*Hint:* First note that $\frac{1}{n!} \leq \frac{1}{2^{n}} for all n\geq 4.$

1. Consider the following statement: “Given $any ϵ>0, a\_{n+1}\_{ ϵ}^{ ≈} a\_{n}$ for n >>1”

Give an *example* of an increasing sequence that satisfies this condition, yet is not a Cauchy sequence.

1. Given a sequence $\left\{a\_{j}\right\} that has the property \left|a\_{n}-a\_{k}\right|\leq \frac{1}{n+k}$ for all *n* and *k*. Prove that $\left\{a\_{n}\right\} $is Cauchy.
2. Given a sequence $\left\{a\_{n}\right\} that has the property \left|a\_{n}-a\_{n+1}\right|\leq \frac{1}{2^{n }} .$ for all *n*. Must it follow that $\left\{a\_{n}\right\} $is Cauchy?

**Exercises:** (S. Abbott, **Understanding Analysis**, 2nd edition, Springer)

1. Give an example of each of the following or argue that no such example exists:
2. A sequence that has a subsequence that is bounded but contains no subsequence that converges.
3. A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these two values.
4. A sequence that contains subsequences converging to every point in the infinite set

{1/2, 1/3, ¼, 1/5, …}.

1. A sequence that contains subsequences converging to every point in the infinite set

{1, 1/2, 1/3, ¼, 1/5, …}, and no subsequences converging to points outside of this set.

1. Decide whether each of the following statements is True or False. Provide either a brief justification or a counterexample.
2. If every proper subsequence of $\left\{x\_{n}\right\}$ converges, then $\left\{x\_{n}\right\}$ converges as well.
3. If $\left\{a\_{n}\right\} contains a divergent subsequence, then \left\{a\_{n}\right\} diverges.$
4. If $\left\{a\_{n}\right\} is bounded and diverges, then there exist two subsequences of \left\{a\_{n}\right\} that converge$

to different limits.

1. If $\left\{a\_{n}\right\} is monotone and contains a convergent subsequence, then \left\{a\_{n}\right\} converges.$
2. Give an example of each of the following or argue that no such example exists:
3. A Cauchy sequence that is not monotone.
4. A Cauchy sequence with an unbounded subsequence.
5. A divergent monotone sequence with a Cauchy subsequence.
6. An unbounded sequence containing a subsequence that is Cauchy.
7. If $\left\{a\_{n}\right\} and \left\{b\_{n}\right\}$ are Cauchy sequences, then one easy way to prove that $\left\{a\_{n}+b\_{n}\right\}$ is Cauchy is to use the Cauchy criterion. Explain!
8. Give a direct argument that $\left\{a\_{n}+b\_{n}\right\} $is Cauchy that does not use the Cauchy criterion.
9. Do the same for the product, $\left\{a\_{n}b\_{n}\right\}$.
10. Let $\left\{a\_{n}\right\} and \left\{b\_{n}\right\}$ be Cauchy sequences. Decide whether or not each of the following is Cauchy, justifying each conclusion.
11. $c\_{n}=\left|a\_{n}-b\_{n}\right|$
12. $c\_{n}=(-1)^{n}a\_{n}$
13. $ c\_{n}=\left⟦a\_{n}\right⟧$ where $\left⟦x\right⟧$ refers to the greatest integer less than or equal to x.

