MATH 351: CLASS DISCUSSION, 5 OCTOBER 2018

1. Complete the proof of the Bolzano-Weierstrass Theorem:

   A *bounded* sequence $\{a_n\}$ *has a convegent subsequence*.

2. Define Cauchy sequence.

3. How are Cauchy sequences related to convergent sequences? What is meant by "Cauchy criterion for convergence"?

4. Show directly that $b_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$ *a Cauchy sequence*.

   Hint: First note that $\frac{1}{n!} \leq \frac{1}{2^n}$ *for all* $n \geq 4$.

5. Consider the following statement: "Given *any* $\epsilon > 0$, $a_{n+1} \underset{\epsilon}{\approx} a_n$ for n >>1"

   Give an *example* of an increasing sequence that satisfies this condition, yet is not a Cauchy sequence.

6. Given a sequence $\{a_j\}$ *that has the property* $|a_n - a_k| \leq \frac{1}{n+k}$ for all *n* and *k*. Prove that $\{a_n\}$ is Cauchy.

7. Given a sequence $\{a_n\}$ *that has the property* $|a_n - a_{n+1}| \leq \frac{1}{2^n}$. for all *n*. Must it follow that $\{a_n\}$ is Cauchy?

Exercises: (S. Abbott, **Understanding Analysis**, 2nd edition, Springer)

1. Give an example of each of the following or argue that no such example exists:
   - (a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
   - (b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these two values.
   - (c) A sequence that contains subsequences converging to every point in the infinite set {1/2, 1/3, ¼, 1/5, …}.
   - (d) A sequence that contains subsequences converging to every point in the infinite set {1, 1/2, 1/3, ¼, 1/5, …}, and no subsequences converging to points outside of this set.
2. Decide whether each of the following statements is True or False. Provide either a brief justification or a counterexample.
   - (a) If every proper subsequence of $\{x_n\}$ converges, then $\{x_n\}$ converges as well.
   - (b) If $\{a_n\}$ contains a divergent subsequence, then $\{a_n\}$ diverges.
   - (c) If $\{a_n\}$ is bounded and diverges, then there exist two subsequences of $\{a_n\}$ that converge to different limits.
   - (d) If $\{a_n\}$ is monotone and contains a convergent subsequence, then $\{a_n\}$ converges.
3. Give an example of each of the following or argue that no such example exists:
   - (a) A Cauchy sequence that is not monotone.
   - (b) A Cauchy sequence with an unbounded subsequence.
   - (c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

4. If $\{a_n\}$ *and* $\{b_n\}$ are Cauchy sequences, then one easy way to prove that $\{a_n + b_n\}$ is Cauchy is to use the Cauchy criterion. Explain!

   (a) Give a direct argument that $\{a_n + b_n\}$ is Cauchy that does not use the Cauchy criterion.

   (b) Do the same for the product, $\{a_n b_n\}$.

5. Let $\{a_n\}$ *and* $\{b_n\}$ be Cauchy sequences. Decide whether or not each of the following is Cauchy, justifying each conclusion.

   (a) $c_n = |a_n - b_n|$

   (b) $c_n = (-1)^n a_n$

   (c) $c_n = [\![a_n]\!]$ where $[\![x]\!]$ refers to the greatest integer less than or equal to x.