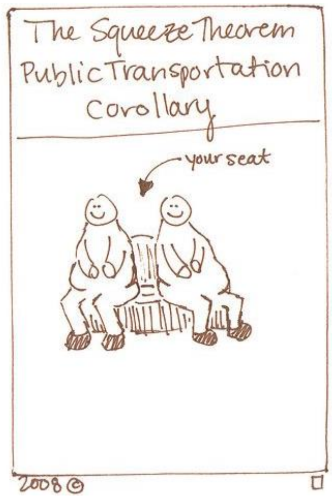
MATH 351: CLASS DISCUSSION, 19 SEPTEMBER

LIMIT THEOREMS



Review:

1. Using the error-form principle, prove that if $a_n \to L, then\ |a_n| \to |L|$. Is the converse true? Proof or counterexample.

2. $If\ a_n = \frac{25n^3+n^2-n+2018}{5n^3-13n^2+1789} = 5$ find an M such that $|e_n| < 0.02$ for all n > M.

3. *Review:* Let $|a| < 1$. For $n \geq 1$, let $S_n = 1 + a + a^2 + \cdots + a^n$.

   Using the error-form principle, prove that $\lim_{n\to\infty} S_n = \frac{1}{1-a}$. *Hint:* Show that $e_n = -\frac{a^{n+1}}{1-a}$.

4. State the three main limit theorems for sequences and prove each using the error-form principle.

5. *Using the limit theorems prove that*

$$\lim_{n\to\infty} \frac{25n^3 + n^2 - n + 2018}{5n^3 - 13n^2 + 1789} = 5$$

6. *(Review of calculus)* Derive a recursive form of Newton's method for finding roots of a differentiable function y = f(x).

7. (a) Applying Newton's method to the polynomial $p(x) = x^2 - 2$, find a recursive sequence that converges to $\sqrt{2}$. Use the error-form principle to prove the result.
   (b) *Practice exercise:* using (a) as a guide, find a recursive sequence that converges to $\sqrt{5}$.

Two more major theorems of the week (in addition to the three limit theorems that you have stated in question 4 above.)

8. Prove the **Squeeze theorem**, *viz:*

   Given three sequences, $\{a_n\},\ \{b_n\},\ \{c_n\}\ satisfying\ \ a_n \leq b_n \leq c_n\ for\ n \gg 1$.

   Suppose that $a_n \to L\ and\ c_n \to L$. $Then\ b_n \to L$.

9. *Example:* Let α be given. Then $\sqrt[n]{2 + \cos n\alpha} \to 1$.

   Hint: Use the result of exercise 3.4/5, *viz.* If c > 0 then $\sqrt[n]{c} \to 1$.

10. *Example:* Prove that $\lim_{n\to\infty} \frac{\ln(n!)}{n \ln n} = 1$. *Hint:* $\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$

11. Find $\lim n(a + \cos n\,\pi)$ *for different values of a.*

12. Prove the **Sequence location theorem**, *viz:*

    $If\ \{a_n\}\ converges$ and $\lim a_n < M$, then $a_n < M$ for $n >> 1$.

13. State the corresponding version of the Sequence location theorem for a convergent sequence bounded below.

14. (a) Prove the **Limit location theorem**, *viz:*

If $\{a_n\}$ *converges and* $\exists M$ *such that* $a_n \leq M$ *for* $n \gg 1$, then $\lim_{n\to\infty} a_n \leq M$.

(b) State the corresponding version of the Limit location theorem for a convergent sequence *bounded below*.

15. Prove the following useful Corollary to the LLT, *viz.* Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and assume that $a_n \leq b_n$ for n >> 1. Then lim a_n = lim b_n.

Exercises from Apostol

In Exercises 1 through 10, compute the limits and explain which limit theorems you are using in each case.

1. $\lim_{x\to 2} \frac{1}{x^2}$.	8. $\lim_{x\to a} \frac{x^2 - a^2}{x^2 + 2ax + a^2}$, $a \neq 0$.
2. $\lim_{x\to 0} \frac{25x3 + 2}{75x7 - 2}$.	9. $\lim_{t\to 0} \tan t$.
3. $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$.	10. $\lim_{t\to 0} (\sin 2t + t^2 \cos 5t)$.
4. $\lim_{x\to 1} \frac{2x^2 - 3x + 1}{x-1}$,	
5 $\lim_{h\to 0} \frac{(t+h)^2 - t^2}{h}$.	
6. $\lim_{x\to 0} \frac{x^2 - a^2}{x^2 + 2ax + a^2}$, $a \neq 0$.	
7. $\lim_{a\to 0} \frac{x^2 - a^2}{x^2 + 2ax + a^2}$, $x \neq 0$.	

22. A function f is defined as follows:

$$f(x) = \begin{cases} \sin x & \text{if } x \leq c, \\ ax + b & \text{if } x > c, \end{cases}$$

where a, b, c are constants. If b and c are given, find all values of a (if any exist) for whichf is continuous at the point x = c.

23. Solve Exercise 22 if f is defined as follows:

$$f(x) = \begin{cases} 2\cos x & \text{if } x \leq c, \\ ax^2 + b & \text{if } x > c. \end{cases}$$

Calculus has its limits.

– Anonymous