MATH 351: CLASS DISCUSSION, 21 SEPTEMBER

LOCATION THEOREMS; SUBSEQUENCES

Review:

1. Using the error-form principle, prove that if $a_n \to L, then\ |a_n| \to |L|$. Is the converse true? Proof or counterexample.

2. *(Review of calculus)* Derive a recursive form of Newton's method for finding roots of a differentiable function y = f(x).

3. (a) Applying Newton's method to the polynomial p(x) = $x^2 - 5$, find a recursive sequence that converges to $\sqrt{5}$. Use the error-form principle to prove the result.

Three more useful theorems of the week (in addition to the three major limit theorems that we have recently studied).

4. Prove the **Squeeze theorem**, *viz:*

Given three sequences, $\{a_n\},\ \{b_n\},\ \{c_n\}\ satisfying\ \ a_n \le b_n \le c_n\ for\ n \gg 1.$

Suppose that $a_n \to L\ and\ c_n \to L$. $Then\ b_n \to L$.

5. Prove: If c > 0 then $\sqrt[n]{c}\ \to 1$. Hint: First consider the case: c > 1.

Then, for the case 0 < c < 1, consider $\frac{1}{c}$.

6. *Example:* Let α be given. Then $\sqrt[n]{2+\cos n\alpha} \to 1$.

7. *Example:* Prove that $\lim_{n\to\infty} \frac{\ln(n!)}{n\ln n} = 1$. *Hint:* $\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$.

Note: This is a simplified version of Sterling's formula.

8. Find $\lim n(a+\cos n\pi)$ *for different values of a.*

9. Prove the **Sequence location theorem**, *viz:*

$If\ \ \{a_n\}\ converges$ and lim a_n < M, then a_n < M for n >> 1.

10. State the corresponding version of the Sequence location theorem for a convergent sequence bounded below.

11. (a) Prove the **Limit location theorem**, *viz:*

$If\ \ \{a_n\}\ converges\ and\ \exists M\ such\ that\ a_n \le M\ for\ n \gg 1$, then $\lim_{n\to\infty} a_n\ \le M$.

(b) State the corresponding version of the Limit location theorem for a convergent sequence *bounded below*.

12. Prove the following useful Corollary to the LLT, *viz.* Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and assume that $a_n \le b_n$ for n >> 1. Then lim $a_n \le$ lim b_n.

13. Using the *Sequence location theorem*, give a more concise proof of the limit theorem for quotients, *viz.*

Given $a_n \to L, L \ne 0, then\ 1/a_n \to 1/L.$

Hint: Consider case 1: L > 0.

14. Define ***subsequence*** of a sequence $\{a_n\}$. Consider the sequence 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, …

Find several convergent subsequences and several divergent sequences.

15. (a) If $\{a_n\}$ is monotone for n>>1, is every subsequence of $\{a_n\}$ monotone for n>>1?

(b) If $\{a_n\}$ is bounded for n>>1, is every subsequence of $\{a_n\}$ bounded for n>>1?

(c) If $a_n \to \infty$, must every subsequence $b_n \to \infty$?

16. State and prove the **Subsequence theorem**.

17. Prove, using the subsequence theorem, that the sequence $\{\sin(\frac{n\pi}{2})\}$ does not converge. Hint: Consider the region where $\sin\frac{n\pi}{2} \geq \frac{\sqrt{2}}{2}$.

18. Prove, using the subsequence theorem, that the sequence $\{\sin n\}$ does not converge.

19. What is meant by a subsequence of a sequence?

20. Prove the Subsequence Theorem: If $\{a_n\}$ converges, then every subsequence also converges, and to the same limit.

21. Prove that $\lim_{n\to\infty} \sin\frac{n\pi}{2}$ *does not exist*. Hint: Consider the region where $\sin\frac{n\pi}{2} \geq \frac{\sqrt{2}}{2}$.

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