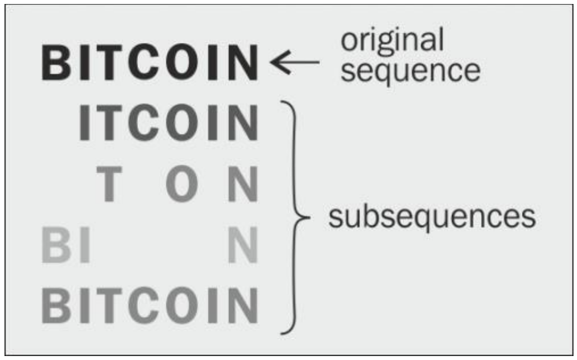
MATH 351: CLASS DISCUSSION, 24 SEPTEMBER

SUBSEQUENCES; NESTED INTERVALS THEOREM



Review:

1. *Example:* Let α be given. Then $\sqrt[n]{2+\cos n\alpha} \to 1$.

2. *Example:* Prove that $\lim\limits_{n\to\infty} \frac{\ln(n!)}{n \ln n} = 1$. *Hint:* $\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$.

Note: This is a simplified version of Sterling's formula.

3. Prove the **Sequence Location Theorem**, *viz:*

$If\ \{a_n\}\ converges$ and $\lim a_n < M$, then $a_n < M$ for $n >> 1$.

4. State the corresponding version of the Sequence location theorem for a convergent sequence bounded below.

5. (a) Prove the **Limit Location Theorem**, *viz:*

$$If\ \{a_n\}\ converges\ and\ \exists M\ such\ that\ a_n \le M\ for\ n \gg 1, \text{then } \lim_{n\to\infty} a_n \le M.$$

(b) State the corresponding version of the Limit location theorem for a convergent sequence *bounded below*.

6. Prove the following useful Corollary to the LLT, *viz.* Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and assume that

$a_n \le b_n$ for $n >> 1$. Then $\lim a_n \le \lim b_n$.

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7. Define ***subsequence*** of a sequence $\{a_n\}$. Consider the sequence 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, …

Find several convergent subsequences and several divergent sequences.

8. (a) If $\{a_n\}$ is monotone for $n>>1$, is every subsequence of $\{a_n\}$ monotone for $n>>1$?

(b) If $\{a_n\}$ is bounded for $n>>1$, is every subsequence of $\{a_n\}$ bounded for $n>>1$?

(c) If $a_n \to \infty$, must every subsequence $b_n \to \infty$?

9. Find an example of a sequence that has countably many subsequences each having a distinct limit.

10. Find an example of a sequence that has uncountably many subsequences, each having a distinct limit.

11. State and prove the **Subsequence Theorem**.

12. Prove, using the subsequence theorem, that the sequence $\{\sin(\frac{n\pi}{2})\}$ does not converge.

13. Prove, using the subsequence theorem, that the sequence $\{\sin n\}$ does not converge.

Hint: Consider the region where $\sin n \geq \frac{\sqrt{2}}{2}$.

14. What is meant by a **nested sequence of intervals**? Give several examples, some of which have intersection (a) containing only one point; (b) containing infinitely many points; (c) empty.

15. State the **Nested Intervals Theorem**. Give a proof. Can any of the hypotheses be eliminated or weakened?

16. Show that, given any $\alpha \in \mathbb{R}$, *there exists a sequence of nested intervals having intersection* $\{\alpha\}$.

17. Example. Let $a_0 = 0$, *and for* $n \geq 1$, *let* $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$
Using the nested interval theorem, prove that $\{a_n\}$ *converges*.

18. Define *cluster point* of a sequence.

19. Find any (and all) cluster points for each of the following sequences:

(a) ½, 1/3, ¼, 1/5, …
(b) 1, 2, 3, 4, 5, …
(c) 1, 0, 1, 0, 1, 0, …
(d) 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, …
(e) 2, 3, 5, 7, 11, 13, …