MATH 351: CLASS DISCUSSION, 26 SEPTEMBER

NESTED INTERVALS THEOREM

1. (a) If $\{a_n\}$ is monotone for n>>1, is every subsequence of $\{a_n\}$ monotone for n>>1?

(b) If $\{a_n\}$ is bounded for n>>1, is every subsequence of $\{a_n\}$ bounded for n>>1?

(c) If $a_n \to \infty$, must every subsequence $b_n \to \infty$?

2. Find an example of a sequence that has countably many subsequences each having a distinct limit.

3. State and prove the **Subsequence Theorem**.

4. Prove, using the subsequence theorem, that the sequence $\{\sin(\frac{n\pi}{2})\}$ does not converge.

5. Prove, using the subsequence theorem, that the sequence $\{\sin n\}$ does not converge.

Hint: Consider the region where $\sin n \geq \frac{\sqrt{2}}{2}$.

6. What is meant by a **nested sequence of intervals**? Give several examples, some of which have intersection (a) containing only one point; (b) containing infinitely many points; (c) empty.

7. State the **Nested Intervals Theorem**. Give a proof. Can any of the hypotheses be eliminated or weakened?

8. Show that, given any $\alpha \in \mathbb{R}$, $there\ exists\ a\ sequence\ of\ nested\ intervals\ having\ intersection\ \{\alpha\}$.

9. Example. Let $a_0 = 0, and\ for\ n \geq 1, let\ a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$
Using the nested interval theorem, prove that $\{a_n\}\ converges$.

10. Definition: K is a ***cluster point*** of $\{a_n\}$ means that, for all $\varepsilon > 0$, there exist infinitely many n for which $|a_n - K| < \varepsilon$.

11. Find any (and all) cluster points for each of the following sequences:

(a) ½, 1/3, ¼, 1/5, …

(b) 1, 2, 3, 4, 5, …
(c) 1, 0, 1, 0, 1, 0, …
(d) 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, …
(e) 2, 3, 5, 7, 11, 13, …