Math 351 solutions: HW I

*solutions to the following:*

* Mattuck, pg. 13 / exercises 1.4.2 and 1.6.3
* Mattuck, pg 14/problem 1-1.
* Apostol, 36/5:

1. Prove that the following sequence converges:



1. Guess a general law that simplifies sn.

**Exercise 1.4.2** Prove the sequence an = nn/n!, n ≥ 1, is (a) increasing; (b) not bounded above (show an ≥ n).

*Solution:*

**part (a)**

Since this is true for all n ≥ 1, we have an+1 > an .

---------------------------------------------------------------------------------

**part (b)**

We begin by proving a lemma that will be useful in the midst of our main proof.

**Lemma:**

**Proof of lemma:** Using the binomial theorem:

Since all terms in this binomial expansion are positive, we easily conclude that

This concludes the Lemma.

- - - - - - - - - - - - - - - -- - - - -

Next, we prove (by induction) the main result: .

For each n ≥ 1, let Hn be the statement: an ≥ n.

*Base case:* When n = 1, a1 = 1. Thus H1 is true.

*Inductive step***:** Let n ≥ 1 be given and assume that Hn is true.

So we assume that an ≥ n.

Now, using the lemma,

Finally, since , {an} is unbounded.

­­­­­­­­­­­­­­­­

**Problem 1.1:** Define a sequence by

where n ≥ 0 and ao is arbitrary.

1. Prove that if ao < 1, the sequence is increasing and bounded above and determine without proof its limit.
2. Consider analogously the case ao > 1.
3. Interpret the sequence geometrically as points on a line; this should make (a) and (b) intuitive.
4. Two proofs here. Assume that ao < 1.

* **Part I:**

To prove: The sequence {an} is bounded above.

For n ≥ 0, let Sn be the statement that an < 1.

* *Base case:* It is given that ao < 1. Hence So, the base case is true.
* *Inductive step:* Let n ≥ 0 be fixed and assume Sn is true. That is, assume that an < 1.

Then= 1. Hence Sn+1 is true.

* **Part II:**

To prove: The sequence {an} is increasing.

Note that for all n, an+1 an = > 0 since, for all n, an < 1 (proven in Part I)

Since the sequence appears to have no upper bound less than 1, we conjecture that the limit of this increasing sequence is 1.

1. Assume that ao > 1. We prove that this is a decreasing sequence bounded below by 1.

* **Part I:**

For n ≥ 0, let Sn be the statement that an > 1.

* *Base case:* It is given that ao > 1. Hence So, the base case is true.
* *Inductive step:* Let n ≥ 0 be fixed and assume Sn is true. That is, assume that an > 1.

Then= 1. Hence Sn+1 is true.

* **Part II:**

To prove: The sequence {an} is decreasing.

Note that for all n ≥ 0, an+1 an = < 0 since for all n ≥ 0, an > 1 (proven in Part I). Hence

{an} is decreasing.

Since the sequence {an} appears to have no lower bound larger than 1, we conjecture that the limit of this decreasing sequence is 1.

1. Assume that ao > 1. Each successive term of the sequence is the average (or midpoint) of the current term and 1. If we write the first few terms, we would see that the sequence {an} is approaching 1 from the right.

Let ao< 1. Each successive term of the sequence is the average (or midpoint) of the current term and one. If we write the first few terms, we would see that the sequence is approaching 1 from the left.

**Apostol question:**

1. Prove that the following sequence converges:



1. Guess a general law that simplifies the product



*Solution:*

1. *Strategy:* We have only to show that {sn} is decreasing and bounded below. Then it follows from the Completeness Property of the real numbers that {sn} converges.

*Proof:* Note that for n ≥ 2 < 1

That is, for all {sn} is decreasing.

Next, since each sn is a product of positive numbers, sn is positive.

And so {sn} is decreasing and bounded below by 0.

Now invoke the Completeness Theorem.

1. We are given, for n ≥ 2,



By calculating the first few terms in the sequence {an}, we conjecture that, for n ≥ 2,



(This result may be proven using induction.)

*It strikes me that mathematical writing is similar to using a language. To be understood you have to follow some grammatical rules. However, in our case, nobody has taken the trouble of writing down the grammar; we get it as a baby does from parents, by imitation of others. Some mathematicians have a good ear; some not (and some prefer the slangy expressions such as 'iff'). That's life.*

- [Jean-Pierre Serre](http://www.college-de-france.fr/default/EN/all/historique/essai.htm)