MATH 351 SOLUTIONS: HW I

solutions to the following:

- Mattuck, pg. 13 / exercises 1.4.2 and 1.6.3
- Mattuck, pg 14/problem 1-1.
- Apostol, 36/5:

  (a) Prove that the following sequence converges:

$$s_n = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

  (b) Guess a general law that simplifies s_n.

Exercise 1.4.2 Prove the sequence $a_n = n^n/n!$, $n \geq 1$, is (a) increasing; (b) not bounded above (show $a_n \geq n$).

Solution:

part (a)

Begin by computing

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{(n+1)!}\ \frac{n!}{n^n} =$$

$$\frac{(n+1)^n\,(n+1)}{n^n}\ \frac{n!}{(n+1)!} = \frac{(n+1)^n\,(n+1)}{n^n}\ \frac{1}{n+1} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n > 1$$

Since this is true for all $n \geq 1$, we have $a_{n+1} > a_n$.

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part (b)

We begin by proving a lemma that will be useful in the midst of our main proof.

Lemma: $\forall n \geq 1\ \left(1+\frac{1}{n}\right)^n \geq 2$

Proof of lemma: Using the binomial theorem:

$$\left(1+\frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \cdots + \frac{1}{n^n}$$

Since all terms in this binomial expansion are positive, we easily conclude that $\left(1+\frac{1}{n}\right)^n \geq 2$.

This concludes the Lemma.

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Next, we prove (by induction) the main result: $\forall n \geq 1 \;\; a_n \;\geq n.$

For each $n \geq 1$, let $\mathcal{H}_n$ be the statement: $a_n \geq n$.

Base case: When $n = 1$, $a_1 = 1$. Thus $\mathcal{H}_1$ is true.

Inductive step: Let $n \geq 1$ be given and assume that $\mathcal{H}_n$ is true.
So we assume that $a_n \geq n$.
Now, using the lemma, $a_{n+1} = \left(1 + \frac{1}{n}\right)^n a_n \;\geq\; 2a_n \geq 2n \;\geq n + 1 \; since \; n \geq 1.$
Finally, since $a_n \geq n \; for \; all \; n$, $\{a_n\}$ is unbounded.

Problem 1.1: Define a sequence by

$$a_{n+1} = \frac{1+a_n}{2} \text{ where } n \geq 0 \text{ and } a_o \text{ is arbitrary.}$$

(a) Prove that if $a_o < 1$, the sequence is increasing and bounded above and determine without proof its limit.
(b) Consider analogously the case $a_o > 1$.
(c) Interpret the sequence geometrically as points on a line; this should make (a) and (b) intuitive.

(a) Two proofs here. Assume that $a_o < 1$.

- **Part I:**

To prove: The sequence $\{a_n\}$ is bounded above.

For $n \geq 0$, let $\mathcal{S}_n$ be the statement that $a_n < 1$.

- *Base case:* It is given that $a_o < 1$. Hence $\mathcal{S}_o$, the base case is true.

- *Inductive step:* Let $n \geq 0$ be fixed and assume $\mathcal{S}_n$ is true. That is, assume that $a_n < 1$.

Then $a_{n+1} = \frac{1+a_n}{2} < \frac{1+1}{2} = 1$. Hence $\mathcal{S}_{n+1}$ is true.

- **Part II:**

To prove: The sequence $\{a_n\}$ is increasing.

Note that for all n, $a_{n+1} - a_n = \frac{1+a_n}{2} - a_n = \frac{1-a_n}{2} > 0$ since, for all n, $a_n < 1$ (proven in Part I)

Since the sequence appears to have no upper bound less than 1, we conjecture that the limit of this increasing sequence is 1.

(b) Assume that $a_o > 1$. We prove that this is a decreasing sequence bounded below by 1.

- **Part I:**

For $n \geq 0$, let S_n be the statement that $a_n > 1$.

- *Base case:* It is given that $a_o > 1$. Hence S_o, the base case is true.

- *Inductive step:* Let $n \geq 0$ be fixed and assume S_n is true. That is, assume that $a_n > 1$.

Then $a_{n+1} = \frac{1+a_n}{2} > \frac{1+1}{2} = 1$. Hence S_{n+1} is true.

- **Part II:**

To prove: The sequence $\{a_n\}$ is decreasing.

Note that for all $n \geq 0$, $a_{n+1} - a_n = \frac{1+a_n}{2} - a_n = \frac{1-a_n}{2} < 0$ since for all $n \geq 0$, $a_n > 1$ (proven in Part I). Hence $\{a_n\}$ is decreasing.

Since the sequence $\{a_n\}$ appears to have no lower bound larger than 1, we conjecture that the limit of this decreasing sequence is 1.

(c) Assume that $a_o > 1$. Each successive term of the sequence is the average (or midpoint) of the current term and 1. If we write the first few terms, we would see that the sequence $\{a_n\}$ is approaching 1 from the right.

Let $a_o < 1$. Each successive term of the sequence is the average (or midpoint) of the current term and one. If we write the first few terms, we would see that the sequence is approaching 1 from the left.

Apostol question:

(a) Prove that the following sequence converges:

$$s_n = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

(b) Guess a general law that simplifies the product

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

Solution:

(a) *Strategy:* We have only to show that $\{s_n\}$ is decreasing and bounded below. Then it follows from the Completeness Property of the real numbers that $\{s_n\}$ converges.

Proof: Note that for $n \geq 2$ $\frac{s_{n+1}}{s_n} = 1 - \frac{1}{(n+1)^2} < 1$
That is, $s_{n+1} < s_n$ for all $n \geq 2, and\ so$ $\{s_n\}$ is decreasing.
Next, since each s_n is a product of positive numbers, s_n is positive.
And so $\{s_n\}$ is decreasing and bounded below by 0.
Now invoke the Completeness Theorem.

(b) We are given, for $n \geq 2$,

$$s_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

By calculating the first few terms in the sequence $\{a_n\}$, we conjecture that, for $n \geq 2$,

$$P(n): \quad a_n = \frac{n+1}{2n}$$

(This result may be proven using induction.)

It strikes me that mathematical writing is similar to using a language. To be understood you have to follow some grammatical rules. However, in our case, nobody has taken the trouble of writing down the grammar; we get it as a baby does from parents, by imitation of others. Some mathematicians have a good ear; some not (and some prefer the slangy expressions such as 'iff'). That's life.

- Jean-Pierre Serre