MATH 351 SOLUTIONS: HW II

Solutions to the following:

Mattuck

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Exercise 2.6.3

Prove that if $\varepsilon > 0$ *is given, then* $\frac{n}{n+2} \underset{\epsilon}{\approx} 1, \ for\ n \gg 1.$

Solution: Choose $N = \frac{2}{\epsilon}$.

Then, for n > N, $\left|\frac{n}{n+2} - 1\right| = \left|\frac{2}{n+2}\right| < \frac{2}{n} < \frac{2}{N} < \epsilon$

Exercise 2.6.4 (b)

Prove that $\{a_n\}$ is decreasing for n>>1, if $\mathbf{a_6} = 1$ and

(b) $a_{n+1} = \frac{n^2+15}{(n+1)(n+2)}\ a_n$

Solution:

$$(b) \quad \frac{a_{n+1}}{a_n} = \frac{n^2+15}{(n+1)(n+2)} < \frac{n^2+15}{(n+1)n} = \frac{1+\frac{15}{n}}{1+n} < 1 \ \ when\, n > 6$$

Problem 2.1

Let $\{a_n\}$ be a sequence. We construct from it another sequence $\{b_n\}$ as follows:

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

(a) Prove that if $\{a_n\}$ is increasing, then $\{b_n\}$ is also increasing.

(b) Prove that if $\{a_n\}$ is bounded above, then $\{b_n\}$ is also bounded above.

Solution:

Part (a): Assume that $\{a_n\}$ is increasing. Then $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$.

Now $b_{n+1} - b_n =$

$$\frac{a_1 + a_2 + \ldots + a_{n+1}}{n+1} - \frac{a_1 + a_2 + \ldots + a_n}{n} =$$

$$\frac{n(a_1 + a_2 + \ldots + a_{n+1}) - (n+1)(a_1 + a_2 + \ldots + a_n)}{n(n+1)} =$$

$$\frac{na_{n+1} - (a_1 + a_2 + \ldots + a_n)}{n(n+1)} =$$

$$\frac{(a_{n+1} - a_1) + (a_{n+1} - a_2) + \ldots + (a_{n+1} - a_n)}{n(n+1)} \geq 0$$

Hence $\{b_n\}$ is increasing.

Part (b): Assume that $\{a_n\}$ is bounded above. Then, by definition, there exists $c \in \mathbf{R}$ such that $a_n \leq c$ for all $n \geq 1$. Thus

$$a_1 + a_2 + \ldots + a_n \leq c + c + \ldots + c = nc$$

and hence, for all $n \geq 1$:

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \leq \frac{nc}{n} = c$$

Thus $\{b_n\}$ is bounded above.

Exercise 3.1.1 (a)

Show that $\lim_{n\to\infty} \frac{\sin n - \cos n}{n} = 0$ directly from the definition of limit.

Solution: Let $\varepsilon > 0$ *be given.*

Then, invoking the triangle inequality,

$$\left|\frac{\sin n - \cos n}{n} - 0\right| = \frac{1}{n}|\sin n - \cos n| \leq \frac{1}{n}(|\sin n| + |\cos n|) \leq \frac{2}{n} < \varepsilon \text{ when } n > \frac{2}{\varepsilon}$$

Exercise 3.2.3 (a)

Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Prove that $a_n \to 0$.

(a) Prove that $\{a_n\}$ converges.

Solution:

Part (a): *The sequence $\{a_n\}$ is bounded above by 1 since:*

$$\boldsymbol{a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \leq \frac{1}{n+1} + \frac{1}{n+1} + \ldots + \frac{1}{n+1} = \frac{n}{n+1} < 1}$$

Next, observe that:

$$a_{n+1} - a_n =$$

$$\left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) =$$

$$-\frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} =$$

$$-\frac{1}{n+1} + \frac{1}{2n+1} + \frac{1/2}{n+1} =$$

$$\frac{1}{2n+1} - \frac{1/2}{n+1} =$$

$$\frac{1/2}{(2n+1)(n+1)} > 0$$

Thus $\{a_n\}$ is strictly increasing.

Now, since $\{a_n\}$ is bounded above and increasing, we invoke the Completeness Theorem to conclude that $\{a_n\}$ converges.

Problem 3.1:

Let $\{a_n\}$ be a sequence. As before, let $\{b_n\}$ be defined as follows:

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

(a) Prove that if $a_n \to 0$, then $b_n \to 0$.

(b) Deduce from part (a) in a few lines that if $a_n \to L$, then $b_n \to L$.

Solution:

Part (a): Assume that $a_n \to 0$.

Let $\varepsilon > 0$ be given. Choose $n^* \in \mathbf{N}$, $n \geq 2$, such that $|a_n - 0| < \varepsilon$ when $n \geq n^*$.

Choose $m^* \in \mathbf{N}$ such

$$m^* \geq \left| \frac{a_1 + a_2 + \ldots + a_{n^+ - 1}}{\varepsilon} \right|$$

Next, let $r^* = \max\{n^*, m^*\}$.

Thus, invoking the Kε-principle, $b_n \to 0$.

Part (b): Assume that $a_n \to L$.

Define $d_n = a_n - L$. We first show that $d_n \to 0$. Let $\varepsilon > 0$ be given. Then a_n is ε-close to L for large n. Using the additivity property, $d_n = a_n - L$ is ε-close to $L - L = 0$. So $d_n \to 0$.

Using part (a), we obtain $y_n \to 0$, where

$$y_n = \frac{d_1 + d_2 + \ldots + d_n}{n}$$

Since

$$\frac{d_1 + d_2 + \ldots + d_n}{n} = \frac{(a_1 - L) + (a_2 - L) + \ldots + (a_n - L)}{n} = \frac{a_1 + a_2 + \ldots + a_n}{n} - L = b_n - L$$

we obtain

$$b_n - L = \frac{a_1 + a_2 + \ldots + a_n}{n} - L \to 0$$

Using the "ε-close"argument given above, we obtain:

$$b_n \to L$$

Problem 3.2: To prove a^n was large if a > 1, we used "Bernoulli's inequality."

$$(1+h)^n \geq 1+nh, \quad \text{if } h \geq 0 .$$

We deduced it from the binomial theorem. This inequality is actually valid for other values of h however. A sketch of the proof starts:

$$\begin{aligned}(1+h)^2 &= 1+2h+h^2 \geq 1+2h, \quad \text{since } h^2 \geq 0 \text{ for all } h;\\ (1+h)^3 &= (1+h)^2(1+h) \geq (1+2h)(1+h), \quad \text{by the previous case,}\\ &= 1+3h+2h^2,\\ &\geq 1+3h .\end{aligned}$$

(a) Show in the same way that the truth of the inequality for the case n implies its truth for the case $n+1$. (This proves the inequality for all n by mathematical induction, since it is trivially true for $n=1$.)

Solution: Assume that h ≥ 0 and that $n \geq 0$.

Let P(n) be the statement that $(1 + h)^n \geq 1 + nh$

Base case: n = 0

LHS = $(1+h)^0 = 1$
RHS = 1 + 0 h = 1
This establishes the base case.

Inductive step: Assume that $(1+h)^n \geq 1+nh$.

Then $(1+h)^{n+1} = (1+h)(1+h)^n \geq (1+h)(1+h)^n \geq (1+h)(1+nh) = 1+nh+nh^2 \geq$

$1+h+nh = 1+(n+1)h$

This proves P(n+1) which completes induction.

(b) For what h is the inequality valid? (Try it when $h = -3$, $n = 5$.) Reconcile this with part (a).

Solution:

If $-1 \leq h \leq 0$, then the induction argument above remains valid.

The key observation is that, $1+h \geq 0 \; when -1 \leq h \leq 0$. And so then the inequality

$(1+h)(1+h)^n \geq (1+h)(1+nh) \; is\ still\ valid\ because\ (1+h)^n \geq 1+nh \geq 0.$