MATH 351 SOLUTIONS: HW III

Solutions to the following:

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Exercise 3.6.1

Prove the following without attempting to evaluate the limt explicitly.

$$\lim_{n\to\infty}\int_1^2 (ln\, x\,)^n\, dx = 0$$

Solution: Observe that, on the interval [1, 2], ln x is non-negative. Also note that ln 2 ≤ ln e = 1.

Since y = ln x is an *increasing* function, $\forall x \in [1,2]\ \ (\ln x)^n \ \le (\ln 2)^n$

Recall that, for 0 < a < 1, $a^n \to 0$.

Next, let $\varepsilon > 0$ *be given. Choose* N *for which* $(\ln 2)^n < \varepsilon$ *for all* $n \ge N$.

Using properties of the Riemann integral:

$$\int_1^2 (ln\, x\,)^n\, dx \le \int_1^2 (ln\, 2\,)^n\, dx = (ln\, 2\,)^n < \varepsilon \text{ for all } n \ge N.$$

Exercise 3.7.1: Show that the sequence $\{a_n\}$, defined below, converges to 0.

$$a_n = \int_0^1 (1-x^2)^n\, dx$$

Solution: *Let $\varepsilon > 0$ be given. Let $\varepsilon^* = \min\{\varepsilon, ½\}$. Define $f_n(x) = (1-x^2)^n$ for $0 \le x \le 1$, and let $b = f(\varepsilon^*)$.*

Notice that f_n is a decreasing non-negative function on [0, 1] with maximum value of 1 at x = 0.

Since $0 < b < 1$, it follows from Theorem 3.4 that $b^n \to 0$.

Choose M such that $|\, b^n - 0| < \varepsilon^*$ when $n \ge M$. Now, using basic properties of the Riemann integral, for n > M:

$$|a_n - 0| = \int_0^1 (1-x^2)^n\,dx = \int_0^{\varepsilon^*} (1-x^2)^n\,dx + \int_{\varepsilon^*}^1 (1-x^2)^n\,dx < (\varepsilon^*)(1) + (1-\varepsilon^*)b^n < \varepsilon^* + (1-\varepsilon^*)\varepsilon^* < 2\varepsilon^* < 2\varepsilon$$

Thus, invoking the Kε-principle, we obtain the desired result: $a_n \to 0$.

Exercise 5.1.4

Given that $a_n/b_n \to L$, $b_n \neq 0$ for all $n \in \mathbf{N}$, and $b_n \to 0$, prove that $a_n \to 0$.

Proof:

Invoking the Product Rule for limits we know that the product of two convergent sequences converges: Thus $a_n = (a_n/b_n)(b_n)$ converges and its limit is the product of the limits of the two convergent sequences: $\lim a_n = \lim (a_n/b_n) \lim b_n = (L)(0) = 0$.

Exercise 5.2.4

$$Let\, a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$$

Prove that $\{a_n\}$ converges and find its limit.

Proof:

We conjecture that $\lim a_n = \ln 2$. To prove this we compare area under the curve $y = 1/x$ from $x = n+1$ to $x = 2n+1$ with upper rectangles of base width 1. This area is smaller than a_n. Hence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} > \int_{n+1}^{2n+1} \frac{1}{x}\,dx = \ln\frac{2n+1}{n+1}$$

Similarly, we compare the area under the curve $y = 1/x$ from $x = n + 1$ to $x = 2n+1$ with lower rectangles of base width 1. This area is smaller than a_n. Thus

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} < \int_{n}^{2n} \frac{1}{x}\,dx = \ln 2$$

Finally:

$$\ln\frac{2n+1}{n+1} < a_n < \ln 2$$

Since, using the laws of limits, $(2n+1)/(n+1) = (2+(1/n))/(1+(1/n)) \to 2$, $\ln((2n+1)/n) \to \ln 2$ and since

(2n)/(n–1) →2, ln((2n)/(n–1))→ln 2. Invoking the Squeeze Theorem, we obtain: $a_n \to \ln 2$.

Problem 5.1 (a) If $a_n \geq 0$ for all $n \in N$ and $a_n \to L$, then $(a_n)^{1/2} \to L^{1/2}$.

Criticize the "proof" given.

Solution: This "proof" assumes that $\lim \sqrt{a_n} = M$ exists. This is a result which must be proven!

Problem 5.1 (b) If $a_n \geq 0$ for all $n \in N$ and $a_n \to L$, then $(a_n)^{1/2} \to L^{1/2}$.

Solution: *Note that the Limit Location Theorem implies that $L \geq 0$; so $L^{1/2}$ is real.*

Case I: *$L \neq 0$*

Let $e_n = (a_n)^{1/2} - L^{1/2}$

Let $\varepsilon > 0$ be given. Then

$$|e_n| = \left|\sqrt{a_n} - \sqrt{L}\right| = \left|\sqrt{a_n} - \sqrt{L}\right| \left(\frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}}\right) = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}} < \varepsilon \quad \text{for } n >> 1$$

Case II: *$L = 0$*

Let $\varepsilon > 0$ be given. Then, since $a_n \to 0$, $|a_n - 0| < \varepsilon^2$ for $n >> 1$.

Hence $(a_n)^{1/2} < \varepsilon$ for $n >> 1$.

Thus $(a_n)^{1/2} \to 0$.
