Math 351 solutions: HW IV

*Solutions to the following:*

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**Exercise 6.4.2**

***Solution:***

 *Proof:* Let {an} be a sequence that has the property:

∃C>0 ∃K∈(0, 1) such | an – an+1| < CKn  for n>>1.

Prove that {an} is a Cauchy sequence.

***Proof:***

*Begin by observing that, for all m>0 and c>0, Km + Km+1 + …+ Km+c < Km/(1 – K).*

*Now, let  > 0 be given. Choose M such that KM < . (This can be done since |K| < 1 ⇒ Kn →0.)*

*According to what is given, ∃ C>0 ∃ K∈(0, 1) ∃ T such that | an – an+1| < CKn  for n ≥ T.*

*Let E = max{M, T}.*

*Next, let i, j ≥ E. Without loss of generality, we may assume that j = i+p for some p>0. Then, for i, j ≥ T:*

*| ai – aj | = |ai+p – ai| = | (ai+p – ai+p-1) + (ai+p-1 – ai+p-2) + (ai+p-2 – ai+p-3) + ... +(ai+1 – ai) | ≤*

*| ai+p – ai+p-1| + |ai+p-1 – ai+p-2| + |ai+p-2 – ai+p-3| + ... + |ai+1 – ai | ≤ CKi+p +CKi+p-1 +CKi+p-2 +... + CKi+1 < C(Ki+1+ Ki+2+ Ki+3+...+ Ki+p) < CKT/(1 – K) < Cε/(1 – K). Using an appropriate version of the K-principle, we have shown that {an} is a Cauchy sequence.*

**Exercise 6.5.4**

***Solution:*** Let S and T be non-empty subsets of **R** and suppose that ∀s∈S ∀t∈T we have s ≤ t.

Prove that sup S ≤ inf T.

*Proof:*

*Let s∈S. Then ∀ t∈T, s ≤ t. Hence s is a lower bound of T, so by definition of infimum, s ≤ inf T. Now this is true for all s∈ S; thus inf T is an upper bound of S. So, by definition of supremum, sup S ≤ inf T.*

**Problem 6.1**

Select two numbers *a* and *b* and let $x\_{0}=a and let x\_{1}=b. $Then continue the sequence by letting each new term be the average of the preceding two:

$x\_{n}=\frac{x\_{n-1}+x\_{n-2}}{2}$ for n ≥ 2.

1. Prove $\left\{x\_{n}\right\} is a Cauchy sequence.$

*Proof:*

 Note that, for all n ≥ 2, xn – xn-1 = $\frac{x\_{n-1}+x\_{n-2}}{2}-x\_{n-1}= \frac{x\_{n-2}- x \_{n-1}}{2}=-\frac{x\_{n-1}- x \_{n-2}}{2}$ (\*)

Hence $\left|x\_{n}-x\_{n-1}\right|=\frac{x\_{n-1}+x\_{n-2}}{2}-x\_{n-1}=\frac{1}{2}\left|x\_{n-1}- x \_{n-2}\right|$ for all n ≥ 2. (\*\*)

At this point, we can use the result of exercise 6.4.2 (which you will prove in the next assignment).

1. Find lim $x\_{n} in terms of a and b.$

*Proof:* Since $\left\{x\_{n}\right\} is a Cauchy sequence, it must converge to a limit L.$

**Case I**: b ≥ a

Letting n ≥ 1, we find:

 $x\_{n}-x\_{0}= \left(x\_{n}-x\_{n-1}\right)+\left( x\_{n-1}-x\_{n-2}\right)+\left(x\_{n-2}-x\_{n-3}\right)+…+\left(x\_{1}-x\_{0}\right)$ (\*\*\*)

Using (\*) in part (a) we can show inductively that $\left(x\_{n}-x\_{n-1}\right) and\left( x\_{n-1}-x\_{n-2}\right) are of opposite sign.$

And using (\*\*), we can show inductively that

$$\left|x\_{n}-x\_{n-1}\right|=\frac{1}{2^{n-1}}\left|x\_{1}- x\_{0}\right|$$

Let  = b – a > 0.

Then using (\*\*\*) and (\*), we have $lim⁡(x\_{n}-a)=\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+…\right)μ=\frac{2}{3} μ$

Hence $lim⁡x\_{n}=a+\frac{2}{3} μ=a+\frac{2}{3}\left(b-a\right)=\frac{1}{3}a+\frac{2}{3}b.$

**Case II:** b < a

Repeating a similar argument to that in case I, we find $lim⁡x\_{n}=\frac{2}{3}a+\frac{1}{3}b$

**Problem 6.5a**

Prove that every sequence {an} has a monotone subsequence.

*Proof: Here we present a clever and concise argument given by Bartle/Sherbert (Introduction to Real Analysis):*

*We say that the mth term of the sequence, an, is a “peak” if am ≥ an for all n ≥ m. Note that in a decreasing sequence, every term is a peak, while in a strictly increasing sequence, no term is a peak.*

***Case I:*** *{an} has infinitely many peaks.*

*We list the peaks by increasing subscripts, *

*Since each term is a peak, we have *

*Therefore the subsequence  of peaks is a decreasing subsequence of {an} .*

***Case II:*** *{an} has a finite number (possibly 0) of peaks.*

*Let these peaks be listed by increasing subscripts:*$ a\_{m\_{1}}, a\_{m\_{2}}, a\_{m\_{3}}, …a\_{m\_{r}}$

$ Let s\_{1}=m\_{r}+1$ *be the first index beyond the last peak.*

*Since* $ a\_{s\_{1}} $*is not a peak, there exists* $s\_{2} > s\_{1} such that a\_{s\_{1}}$*<*$ a\_{s\_{2}}$*.*

*Since* $ a\_{s\_{2}} $*is not a peak, there exists* $s\_{3}> s\_{2} such that a\_{s\_{2}}$*<*$ a\_{s\_{3}}$*.*

*Since* $ a\_{s\_{3}}$*is not a peak, there exists* $s\_{4}> s\_{3} such that a\_{s\_{3}}$*<*$ a\_{s\_{4}}$*.*

*Continuing in this way, we obtain a (strictly) increasing subsequence*$ \left\{a\_{s\_{j}}\right\} of \left\{a\_{n}\right\}$*.*