MATH 351 SOLUTIONS: HW IV

Solutions to the following:

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Exercise 6.4.2

Solution:

Proof: Let $\{a_n\}$ be a sequence that has the property:

$\exists C>0 \ \ \exists K \in (0, 1)$ such $|a_n - a_{n+1}| < CK^n$ for $n >> 1$.

Prove that $\{a_n\}$ is a Cauchy sequence.

Proof:

Begin by observing that, for all $m>0$ and $c>0$, $K^m + K^{m+1} + \ldots + K^{m+c} < K^m/(1 - K)$.

Now, let $\varepsilon > 0$ be given. Choose M such that $K^M < \varepsilon$. (This can be done since $|K| < 1 \Rightarrow K^n \to 0$.)

According to what is given, $\exists C>0 \ \exists K \in (0, 1) \ \exists T$ such that $|a_n - a_{n+1}| < CK^n$ for $n \geq T$.

Let $E = \max\{M, T\}$.

Next, let $i, j \geq E$. Without loss of generality, we may assume that $j = i+p$ for some $p>0$. Then, for $i, j \geq T$:

$|a_i - a_j| = |a_{i+p} - a_i| = |(a_{i+p} - a_{i+p-1}) + (a_{i+p-1} - a_{i+p-2}) + (a_{i+p-2} - a_{i+p-3}) + \ldots + (a_{i+1} - a_i)| \leq$

$|a_{i+p} - a_{i+p-1}| + |a_{i+p-1} - a_{i+p-2}| + |a_{i+p-2} - a_{i+p-3}| + \ldots + |a_{i+1} - a_i| \leq CK^{i+p} + CK^{i+p-1} + CK^{i+p-2} + \ldots + CK^{i+1} <$

$C(K^{i+1} + K^{i+2} + K^{i+3} + \ldots + K^{i+p}) < CK^T/(1 - K) < C\varepsilon/(1 - K)$. Using an appropriate version of the $K\varepsilon$-principle, we have shown that $\{a_n\}$ is a Cauchy sequence.

Exercise 6.5.4

Solution: Let S and T be non-empty subsets of **R** and suppose that $\forall s \in S\ \forall t \in T$ we have $s \leq t$.

Prove that $\sup S \leq \inf T$.

Proof:

Let $s \in S$. Then $\forall t \in T$, $s \leq t$. Hence s is a lower bound of T, so by definition of infimum, $s \leq \inf T$. Now this is true for all $s \in S$; thus inf T is an upper bound of S. So, by definition of supremum, $\sup S \leq \inf T$.

Problem 6.1

Select two numbers a and b and let $x_0 = a \text{ and let } x_1 = b$. Then continue the sequence by letting each new term be the average of the preceding two:

$x_n = \frac{x_{n-1}+x_{n-2}}{2}$ for n ≥ 2.

(a) Prove $\{x_n\}$ *is a Cauchy sequence.*

Proof:

Note that, for all n ≥ 2, $x_n - x_{n-1} = \frac{x_{n-1}+x_{n-2}}{2} - x_{n-1} = \frac{x_{n-2} - x_{n-1}}{2} = -\frac{x_{n-1} - x_{n-2}}{2}$ (*)

Hence $|x_n - x_{n-1}| = \frac{x_{n-1}+x_{n-2}}{2} - x_{n-1} = \frac{1}{2}|x_{n-1} - x_{n-2}|$ for all n ≥ 2. (**)

At this point, we can use the result of exercise 6.4.2 (which you will prove in the next assignment).

(b) Find lim x_n *in terms of a and b.*

Proof: Since $\{x_n\}$ *is a Cauchy sequence, it must converge to a limit L.*

Case I: b ≥ a

Letting n ≥ 1, we find:

$x_n - x_0 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + (x_{n-2} - x_{n-3}) + \cdots + (x_1 - x_0)$ (***)

Using (*) in part (a) we can show inductively that $(x_n - x_{n-1})$ *and* $(x_{n-1} - x_{n-2})$ *are of opposite sign.*

And using (**), we can show inductively that

$$|x_n - x_{n-1}| = \frac{1}{2^{n-1}}|x_1 - x_0|$$

Let μ = b – a > 0.

Then using (***) and (*), we have $\lim(x_n - a) = \left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\right)\mu = \frac{2}{3}\mu$

Hence $\lim x_n = a + \frac{2}{3}\mu = a + \frac{2}{3}(b-a) = \frac{1}{3}a + \frac{2}{3}b.$

Case II: b < a

Repeating a similar argument to that in case I, we find $\lim x_n = \frac{2}{3}a + \frac{1}{3}b$

Problem 6.5a

Prove that every sequence $\{a_n\}$ has a monotone subsequence.

Proof: *Here we present a clever and concise argument given by Bartle/Sherbert (Introduction to Real Analysis):*

We say that the m^{th} term of the sequence, a_n, is a "peak" if $a_m \geq a_n$ for all $n \geq m$. Note that in a decreasing sequence, every term is a peak, while in a strictly increasing sequence, no term is a peak.

Case I: *$\{a_n\}$ has infinitely many peaks.*

We list the peaks by increasing subscripts, $a_{m_1}, a_{m_2}, a_{m_3}, a_{m_4}, a_{m_5}, \ldots$

Since each term is a peak, we have $a_{m_1} \geq a_{m_2} \geq a_{m_3} \geq a_{m_4} \geq a_{m_5} \geq \ldots$

Therefore the subsequence $\{a_{m_k}\}$ *of peaks is a decreasing subsequence of* $\{a_n\}$.

Case II: *$\{a_n\}$ has a finite number (possibly 0) of peaks.*

Let these peaks be listed by increasing subscripts: $a_{m_1}, a_{m_2}, a_{m_3}, \ldots a_{m_r}$

Let $s_1 = m_r + 1$ *be the first index beyond the last peak.*

Since a_{s_1} *is not a peak, there exists* $s_2 > s_1$ *such that* $a_{s_1} < a_{s_2}$.

Since a_{s_2} *is not a peak, there exists* $s_3 > s_2$ *such that* $a_{s_2} < a_{s_3}$.

Since a_{s_3} *is not a peak, there exists* $s_4 > s_3$ *such that* $a_{s_3} < a_{s_4}$.

Continuing in this way, we obtain a (strictly) increasing subsequence $\{a_{s_j}\}$ *of* $\{a_n\}$.