MATH 351 PRACTICE TEST II

PART A

Definitions and statements of theorems.

PART B

Each of the following assertions is false. Give an explicit counter-example to illustrate this.

1. Every sequence has at least one convergent subsequence.
2. Let (a_n) be a bounded sequence and let c be a constant. Then:

$$\limsup ca_n = c \limsup a_n.$$

3. Let (a_n) be a Cauchy sequence of irrational numbers. Then $\lim a_n$ is irrational.
4. Let $\{c_n\}$ be a subsequence of $\{a_n\}$ that is strictly increasing and let $\{b_n\}$ be a subsequence of $\{a_n\}$ that is strictly decreasing. Then $\{a_n\}$ is divergent.
5. Suppose that the sequence $\left\{\frac{a_n}{n}\right\}$ converges. Then $\{a_n\}$ converges.
6. Let $\{a_n\}$ be a sequence. Suppose that $|a_{n+2} - a_{n+1}| < |a_{n+1} - a_n|$ for all $n \in \mathbf{N}$. Then $\{a_n\}$ is a Cauchy sequence.

7. If $\{a_n\}$ diverges then the sequence $\{\sin(a_n)\}$ must diverge.

8. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences satisfying $0 \le a_n \le b_n$ for all $n \in \mathbf{N}$. Then if $\{a_n\}$ diverges it must follow that $\{b_n\}$ diverges.

9. If two positive series $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ diverges.

10. If $\sum_{n=1}^{\infty} a_n$ is a convergent positive series, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

11. A sequence $\{a_n\}$ cannot have a countable number of cluster points.

12. If $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

13. If the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ each diverge then $\sum_{n=0}^{\infty} a_n b_n$ diverges.

14. Let *A* and *B* be non-empty bounded sets. Then $A \cup B$ is bounded and

    $\sup (A \cup B) = \sup A + \sup B$.

15. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences.

    Then $\{a_n b_n\}$ is bounded and $\limsup a_n b_n = (\limsup a_n)(\limsup b_n)$.

16. Let *A* be a non-empty bounded set. Then $\inf A < \sup A$.

17. Let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying $a_n < b_n$ for all $n \ge 0$ and $b_n - a_n \to 0$.

    Let $I_n = (a_n, b_n)$. Then $\bigcap_{n=0}^{\infty} I_n$ consists of a single point.

PART C

Instructions: Select any 3 of the following 4 problems. You may answer all 4 to obtain extra credit.

1. A sequence has the property that $|a_n - a_{n+1}| \le 1/n!$ for $n >> 1$. Prove that $\{a_n\}$ is Cauchy.

2. Consider the proof of the theorem that *every Cauchy sequence $\{a_n\}$ converges*:

First, one shows that $\{a_n\}$ is bounded. Secondly, one shows that there exists a convergent subsequence $\{a_{n_j}\}$ *of* $\{a_n\}$. Let $L = \lim a_{n_j}$.
Finally one must show that $\{a_n\}$ converges to *L*.

Give the *proof of this last result*.

3. Given a conditionally convergent series, $\sum_{n=0}^{\infty} b_n$, prove that the series, $\sum_{n=0}^{\infty} b_n^{+}$, formed from its positive terms, diverges.

4. Consider the proof of the Completeness Property for sets of real numbers:

> Let *S* be non-empty and bounded above.
> We construct a sequence of nested intervals $I_n = [a_n, b_n]$ satisfying: I_n contains at least one point of *S*, b_n is an upper bound of *S* and $b_n - a_n \to 0$.
> Invoking the Nested Intervals Theorem, there exists a point *c* such that $\lim a_n = c$ and $\lim b_n = c$.

Prove that c = sup S.

Part D

Instructions: Select any 4 of the following 6 problems. You may answer 5 to obtain extra credit but do not try to solve all 6. To receive credit, you must show your work!

1. Let *A* be *B* be two bounded and non-empty subsets of R.

Define $A + B = \{a+b \mid a \in A, b \in B\}$. Prove that $\sup(A + B) \leq \sup A + \sup B$.

2. For each of the two given series, investigate convergence or divergence:

$$(a) \quad \sum_{1}^{\infty} \sin\left(\frac{1+n}{2+3n^2 \ln n}\right)$$

$$(b) \quad \sum_{2}^{\infty} \frac{1}{(\ln n)^n}$$

3. For each of the two given series, investigate convergence (conditional or absolute) or divergence:

$$(a) \quad \sum_{1}^{\infty} (-1)^n \frac{3^n (n!)^2}{(2n)!}$$

$$(b) \quad \sum_{2}^{\infty} \frac{(-1)^n}{(\ln n)^{1/n}}$$

4. Let $\{a_n\}$ and $\{b_n\}$ be positive sequences and assume that $\sum (a_n)^2$ and $\sum (b_n)^2$ each converge.

Prove that $\sum a_n b_n$ converges. (*Hint:* Use the Comparison Test.)

5. Let $\sum_{1}^{\infty} a_n$ be a convergent positive series. Prove that $\sum_{1}^{\infty} \sqrt{a_n a_{n+1}}$ converges also.

Hint: Use the geometric-mean arithmetic mean inequality.

6. Let S be a non-empty bounded set of real numbers and let $\beta = \sup S$. Prove that

$\inf\{ \beta - x \mid x \in S\} = 0$