MATH 351 OLD TEST 1 FEBRUARY 2018

PART A: Definitions *[10 points each]* Be precise and careful.

1. State the *Completeness property* of the real numbers

2. Define *cluster point* of a sequence.

3. State the *Nested Intervals theorem*.

4. Define $\lim_{n\to\infty} a_n = L$.

5. State the *Limit Location Theorem.*

PART B: ***True or False*** *[5 points each]*

Determine if each of the following statements is *True* or *False*. If False, provide a *precise counter-example*; if True, give a *very brief* justification.

(a) If $\{a_n\}$ is bounded above, then the sequence $\{b_n\}$ defined by $b_n = -a_n$ is bounded below.

(b) If $\{a_n\}$ and $\{b_n\}$ are bounded below, then $\{a_n b_n\}$ is bounded below.

(c) If $\{a_n^2\}$ converges to 9 and $\{a_n^3\}$ converges to 27, then $\{a_n\}$ converges to 3.

(d) If $\{a_n + b_n\}$ converges and $\{a_n - b_n\}$ converges, then $\{a_n\}$ converges.

(e) If $\{a_n b_n\}$ converges and $\{b_n\}$ converges, then $\{a_n\}$ converges.

(f) If $\{|a_n|\}$ converges then $\{a_n\}$ converges.

(g) If $\{a_n\}$ is decreasing and $\{b_n\}$ is decreasing, then $\{a_n b_n\}$ is decreasing.

(h) Consider the sequence, $\{a_n\}$, defined recursively:

$$a_1 = 2$$
$$a_{n+1} = \sqrt{a_n + 4} \quad for\ all\ n \geq 1.$$

Then $a_n \leq 3$ for all $n \geq 1$.

(i) Every sequence has at least one convergent subsequence.

(j) If $\{a_n + b_n\}$ converges and $\{a_n\}$ diverges then $\{b_n\}$ converges.

(k) Let $\{a_n\}$ be a sequence satisfying $a_n > 5\ for\ n \gg 1$.
If $\{a_n\}$ converges to L, then L > 5.

(l) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences satisfying $0 \le a_n \le b_n$ for all $n \in \mathbf{N}$. Then, if $\{a_n\}$ diverges, it follows that $\{b_n\}$ diverges.

(m) Let $\{c_n\}$ be a subsequence of $\{a_n\}$ that is *strictly* increasing and let $\{b_n\}$ be a subsequence of $\{a_n\}$ that is *strictly* decreasing. Then $\{a_n\}$ is divergent.

(n) Suppose that the sequence $\{a_n / n\}$ converges. Then $\{a_n\}$ converges.

PART C: PROOFS *[12 points each]*

Instructions: Select any 4 of the following 6 problems. You may answer more than 4 to earn extra credit.

1. Prove that every convergent sequence is bounded.

2. State and prove the *Sequence Location Theorem.*

3. For $n \ge 1$, define the sequence $\{c_n\}$ as follows:

$$c_n = \int_0^1 e^{-\frac{x^3}{n}}\, dx$$

Determine $L = \lim c_n$. Prove that $\{c_n\}$ converges to L.

[*Advice:* Sketch the integrand for several values of *n*.]

4. State and prove the *Product Theorem for Limits*.

Advice: Use the Error-form Principle.

5. Determine $\lim \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right)$ and prove it.

Hint: Use an area argument to estimate the sum of

$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$, followed by the Squeeze Theorem.

6. Assume that $a_n > 0 \; for\ all\ n,\ \ and\ that\ \frac{a_{n+1}}{a_n} \to L\ where\ L < 1$.

(a) Explain why L ≥ 0.

(b) Prove that $\{a_n\}\ is\ decreasing$ for n>>1.

(c) Prove that $a_n \to 0$.